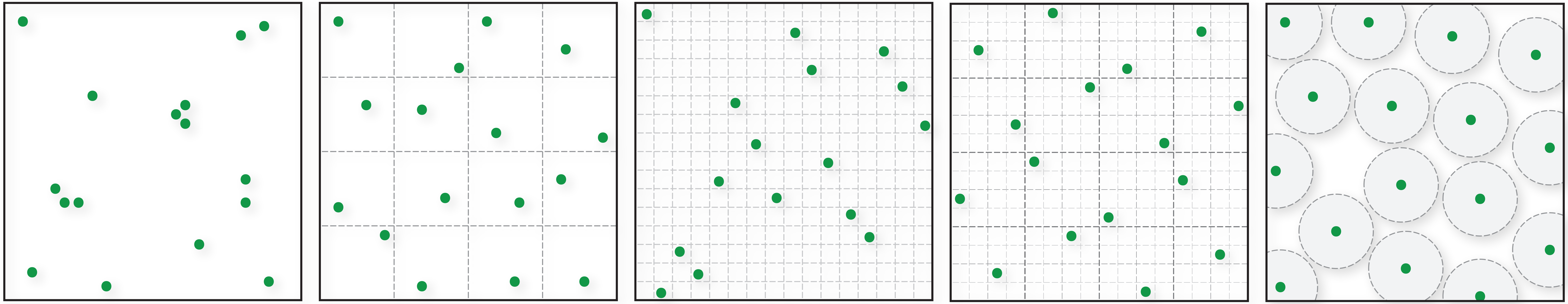
Popular Sampling Patterns

Fourier Analysis of Numerical Integration in Monte Carlo Rendering



Wojciech Jarosz

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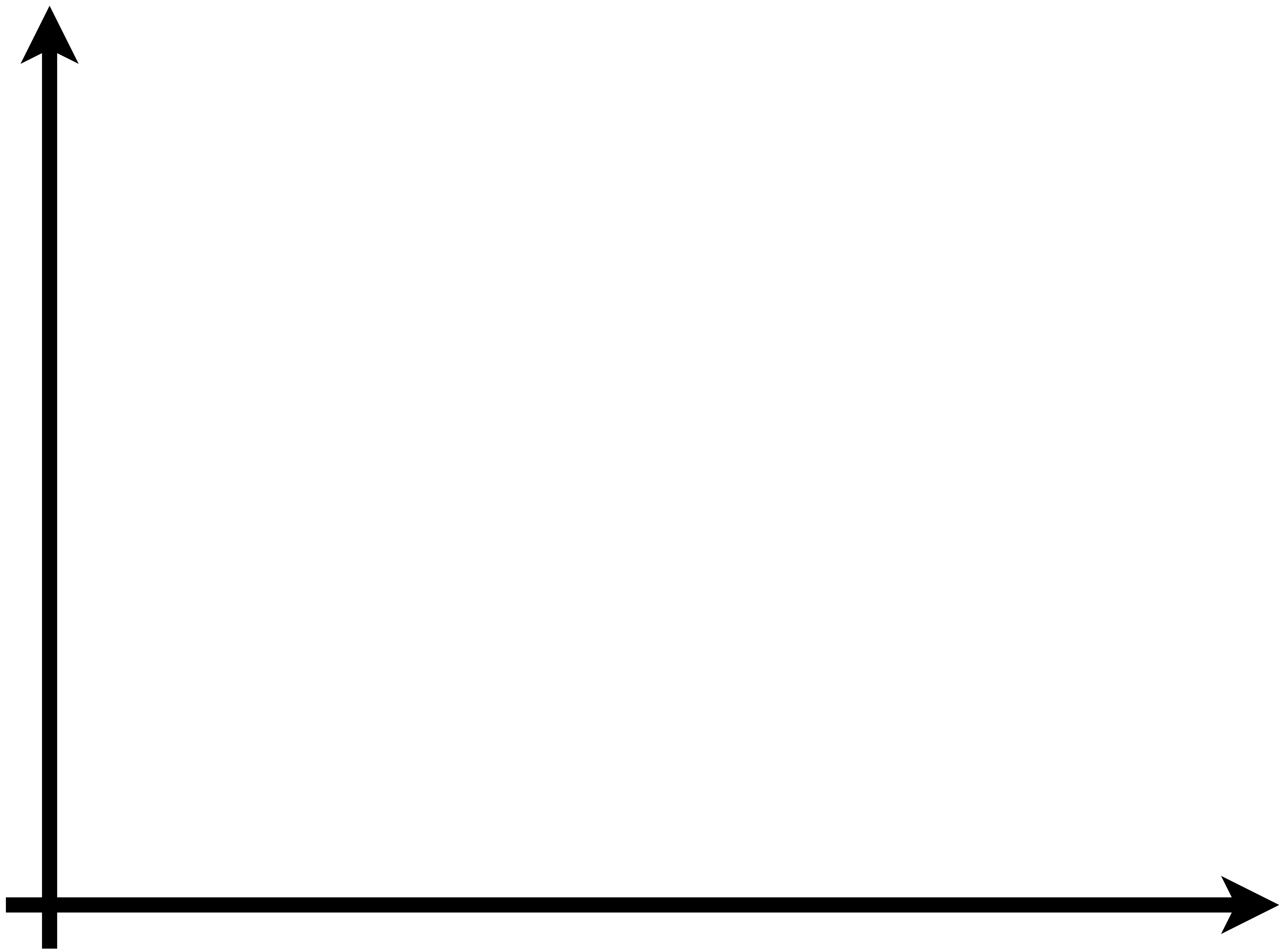
DARTMOUTH
VISUAL COMPUTING LAB

Render the Possibilities
SIGGRAPH2016

Recall: Monte Carlo Integration

$$I = \int_D f(x)\,\mathrm{d}x$$



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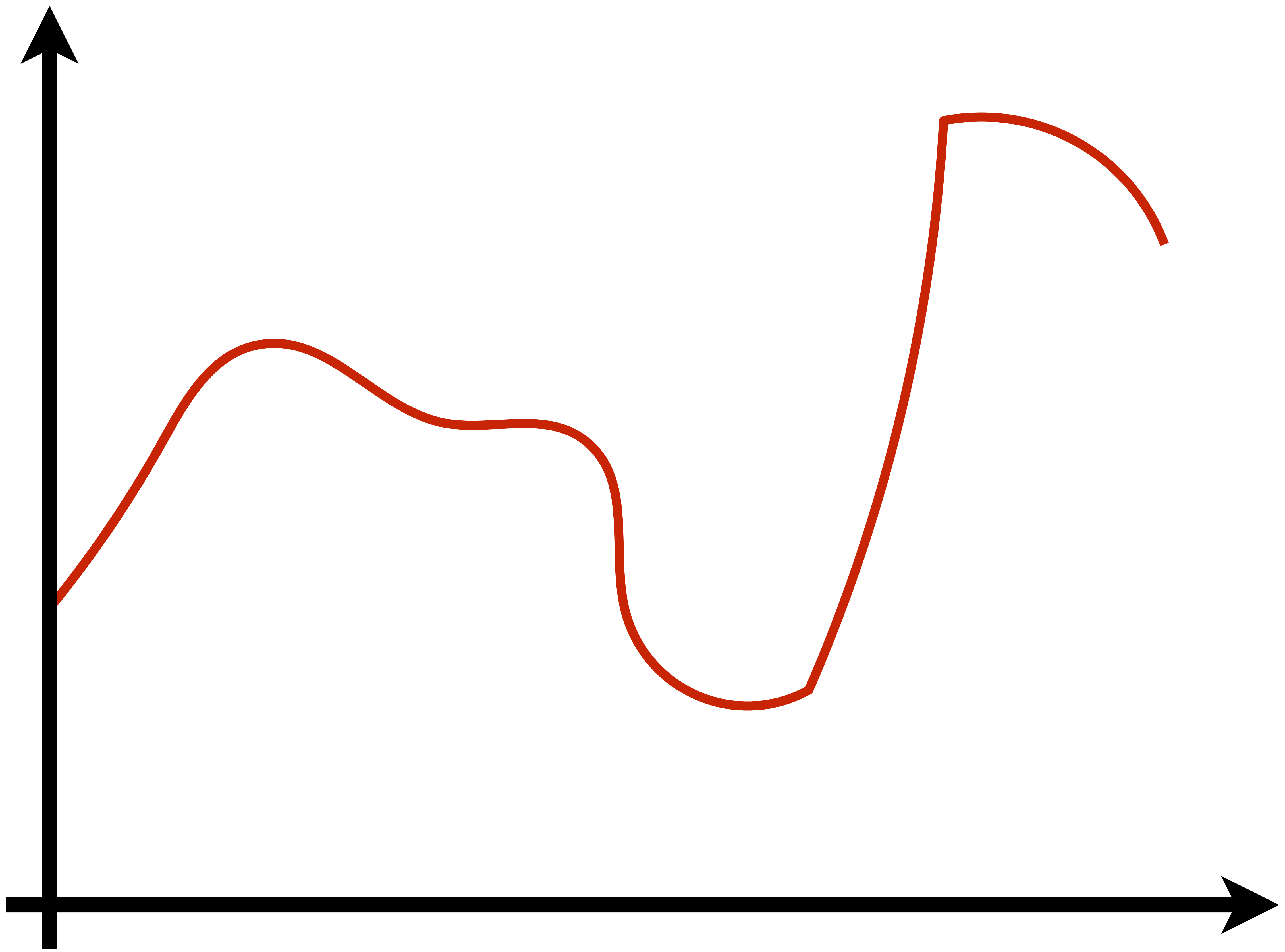

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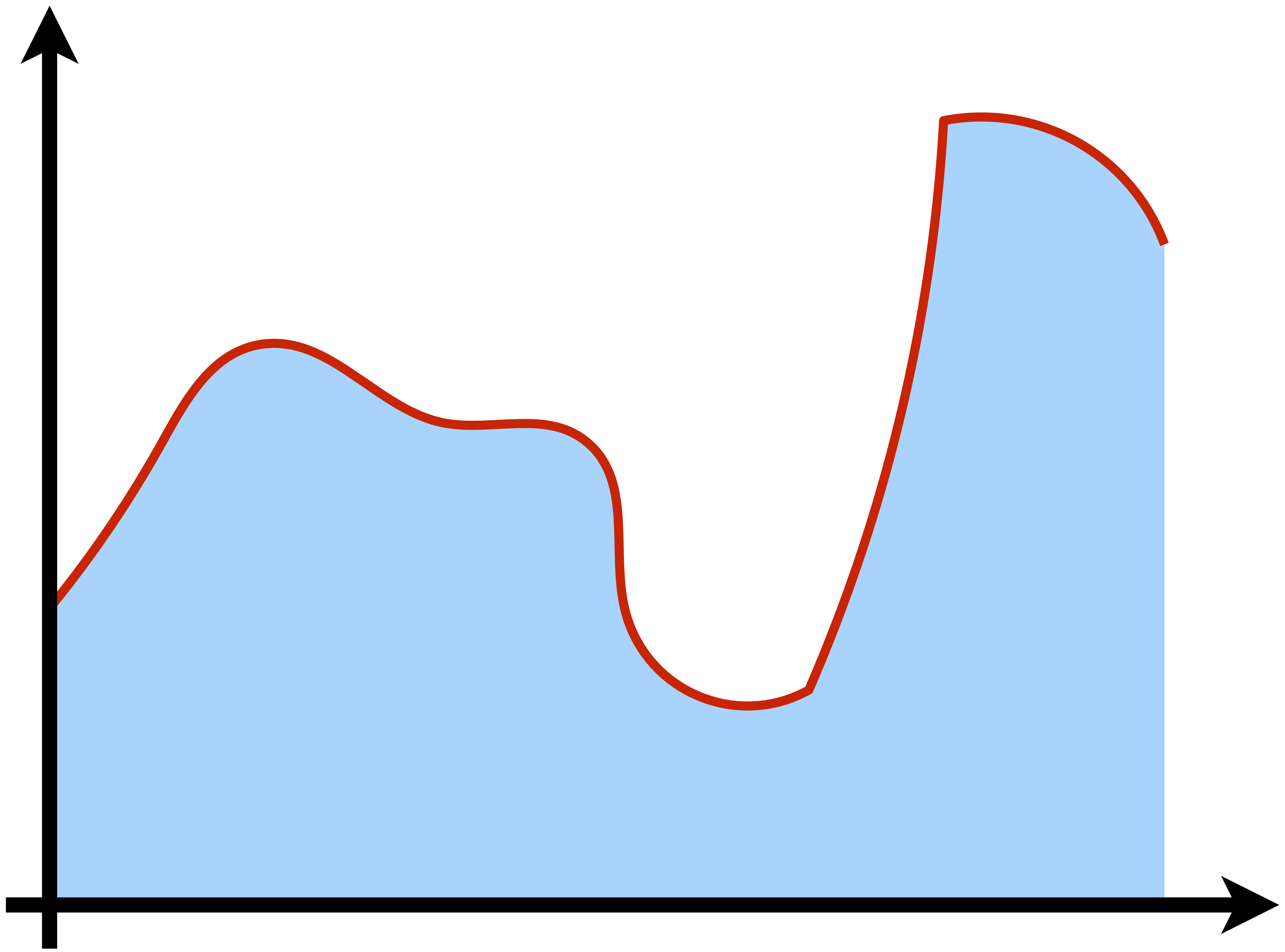

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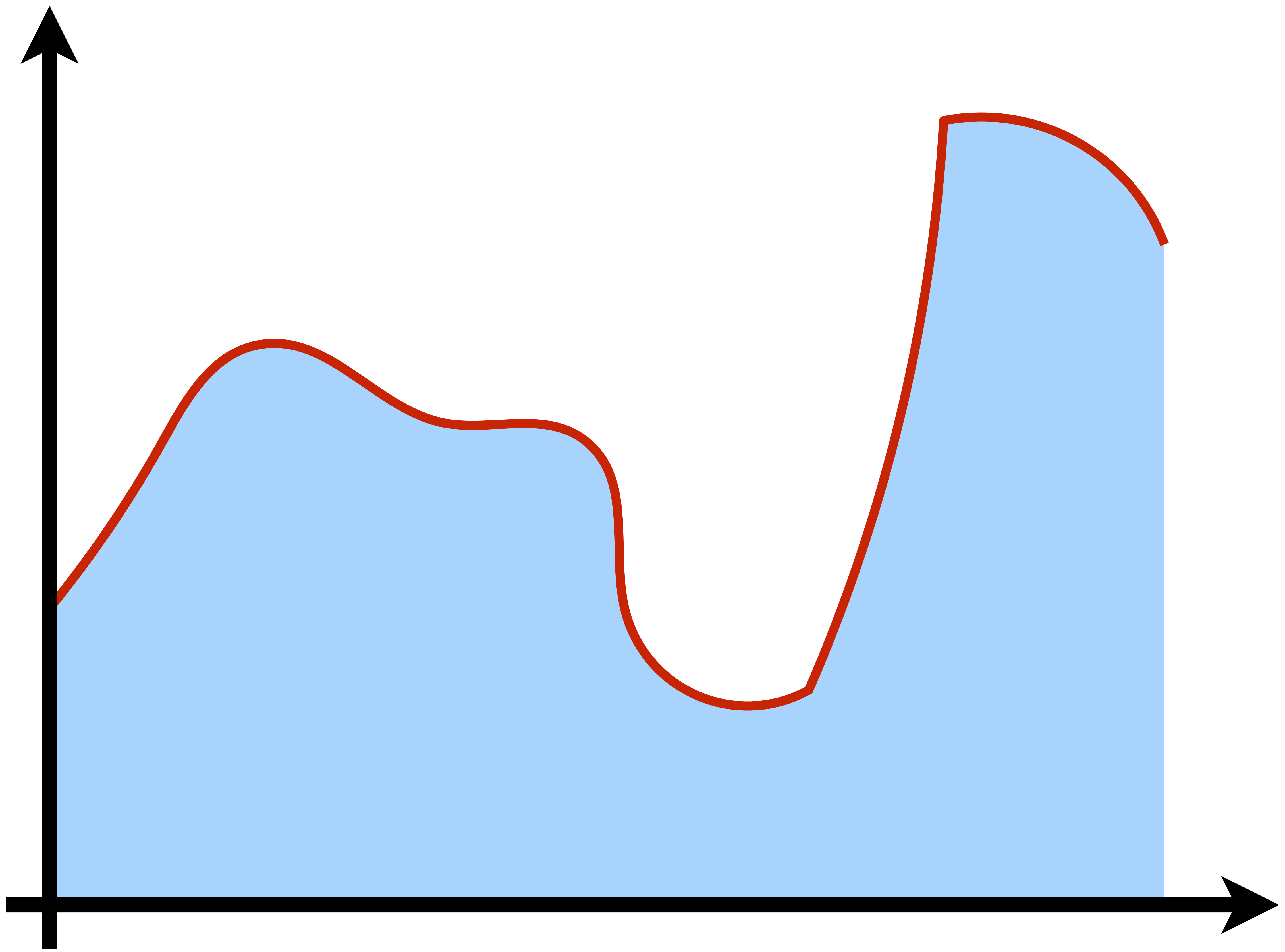
$$\approx \int_D f(x)\,\mathbf{S}(x)\,\mathrm{d}x$$



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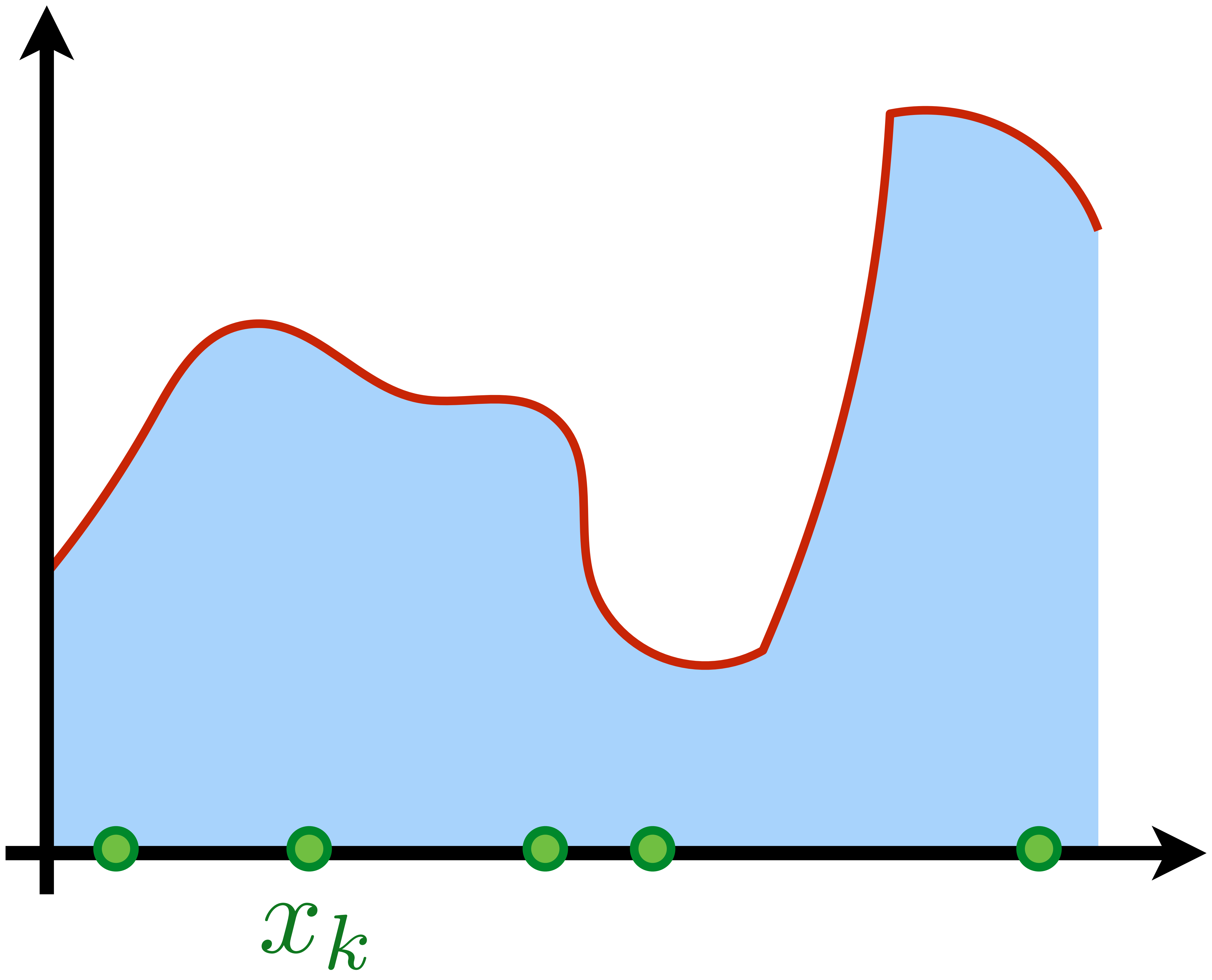
$$\mathbf{S}(x) = \frac{1}{N} \sum_{k=1}^{N} \delta(x - x_k)$$



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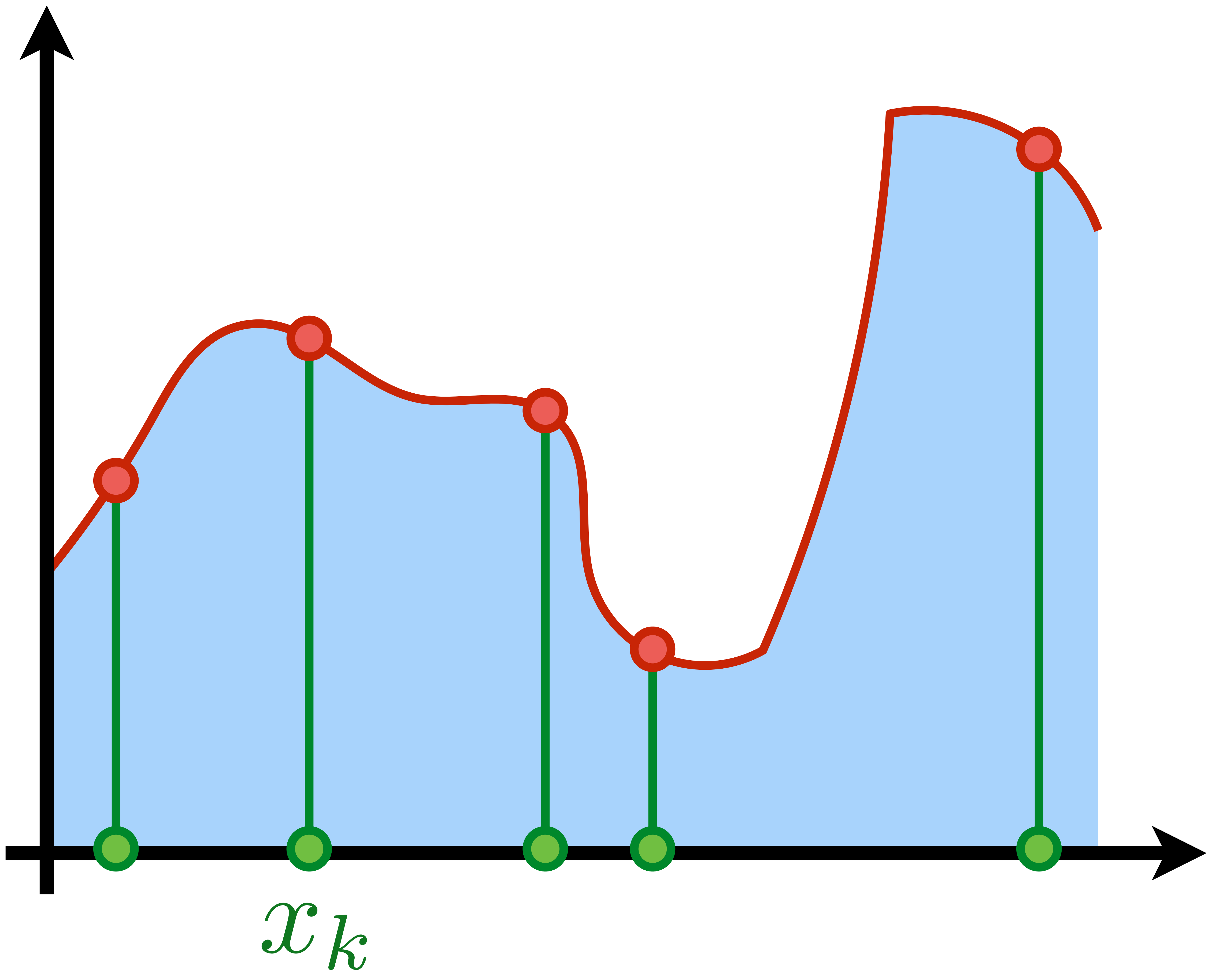

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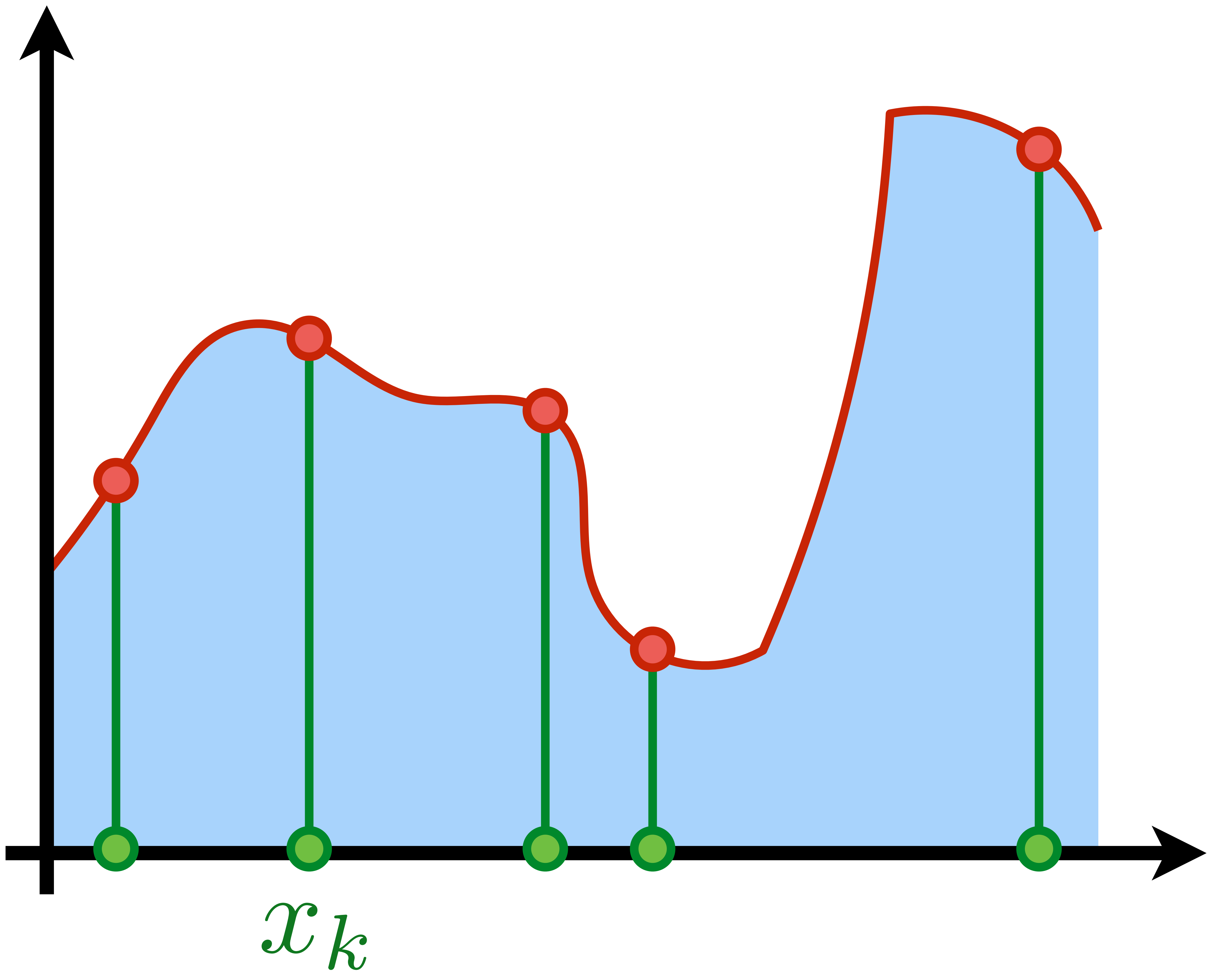
How to generate the locations x_k?



Independent Random Sampling

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for (int k = 0; k < num; k++)
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    samples(k).x = randf();
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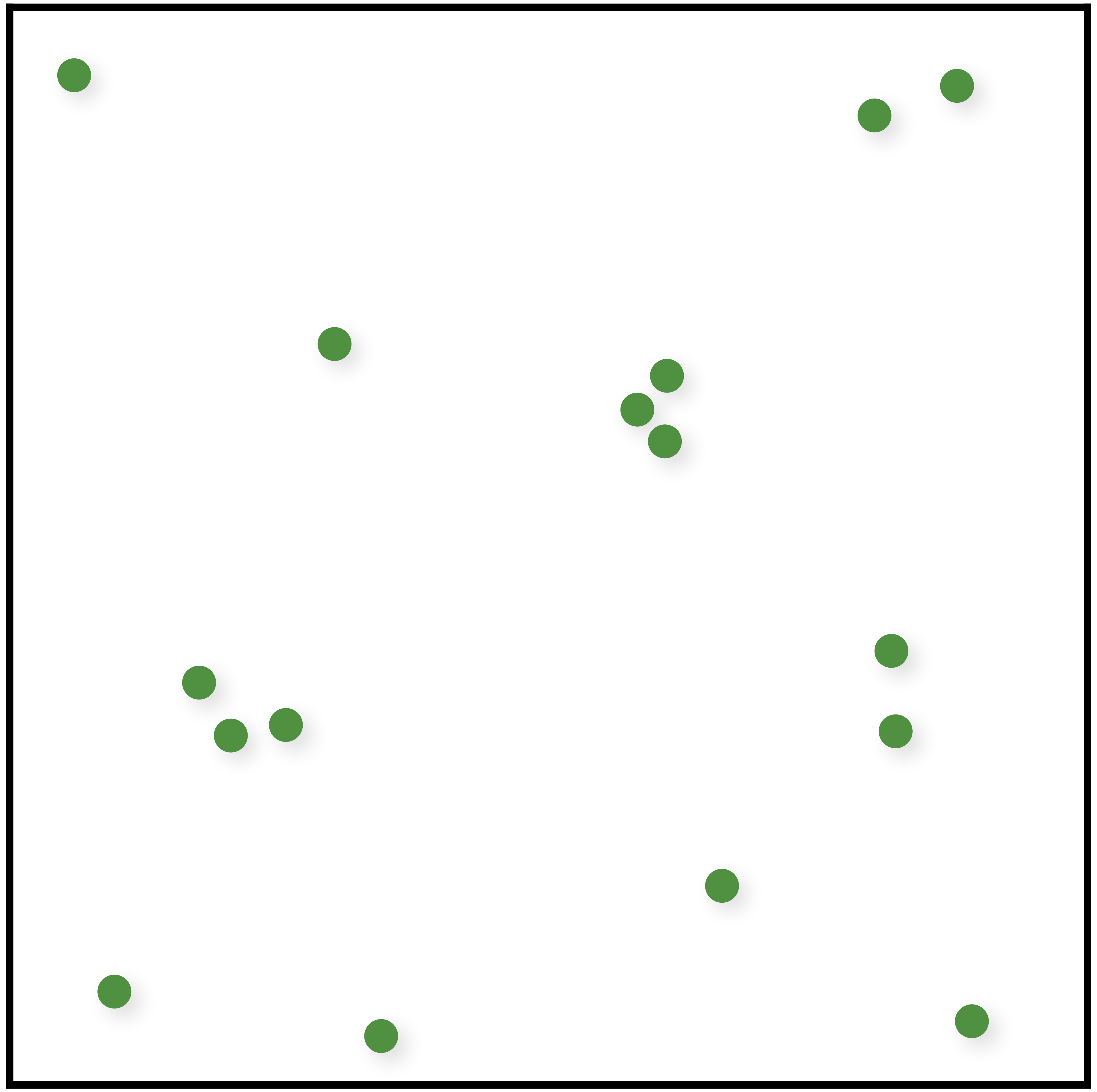

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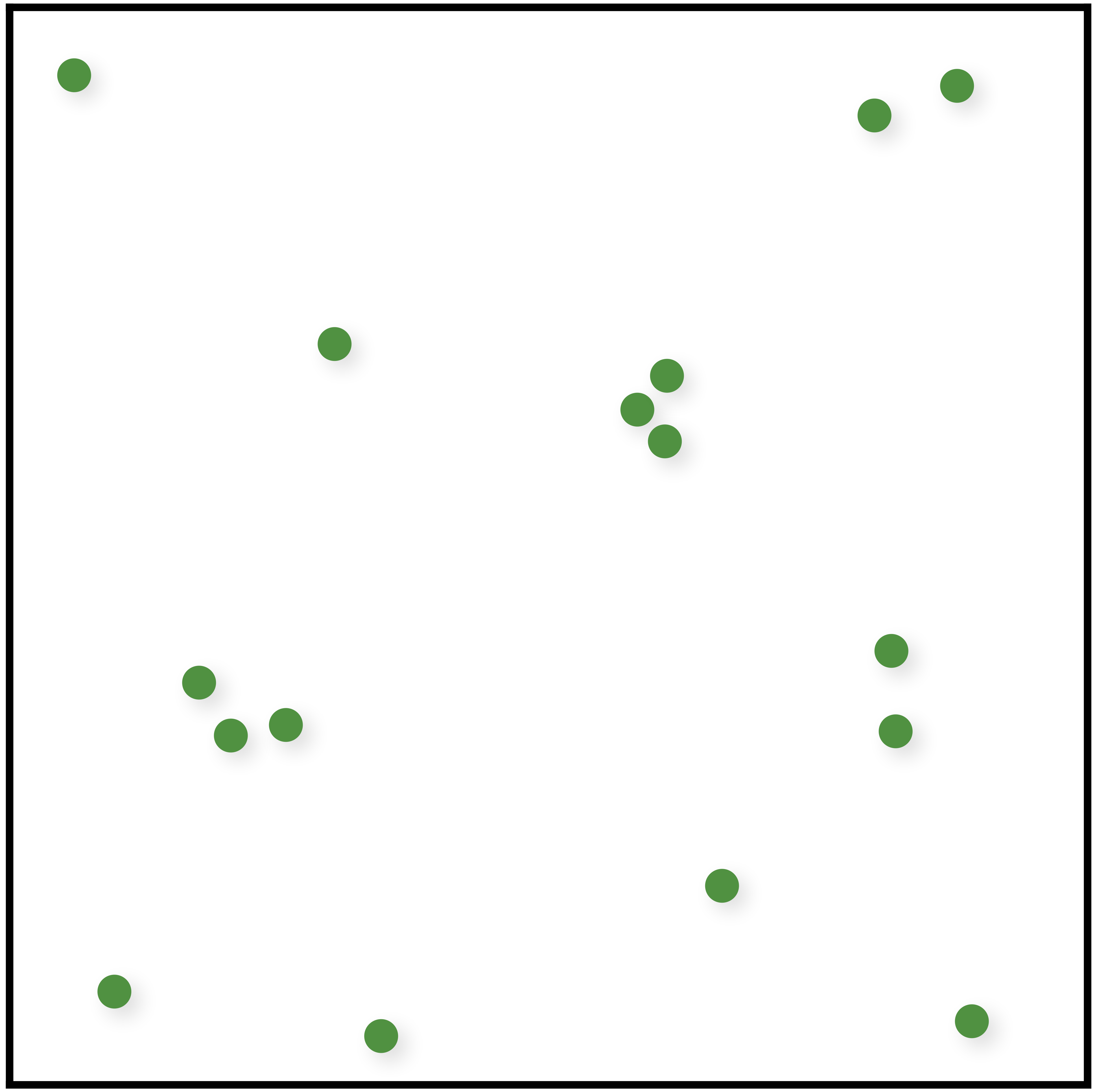
✔ Trivially extends to higher dimensions



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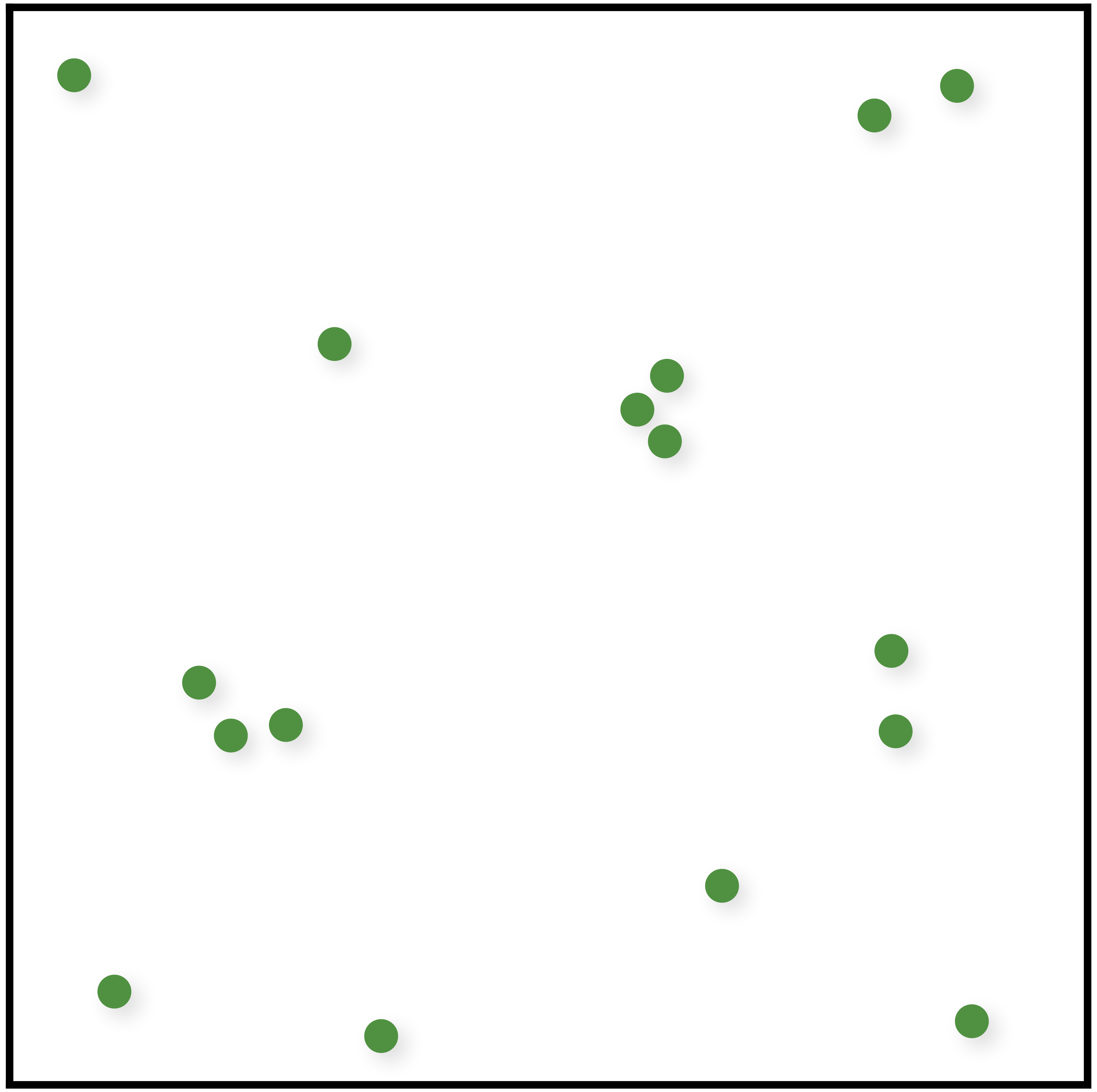
✔ Trivially progressive and memory-less



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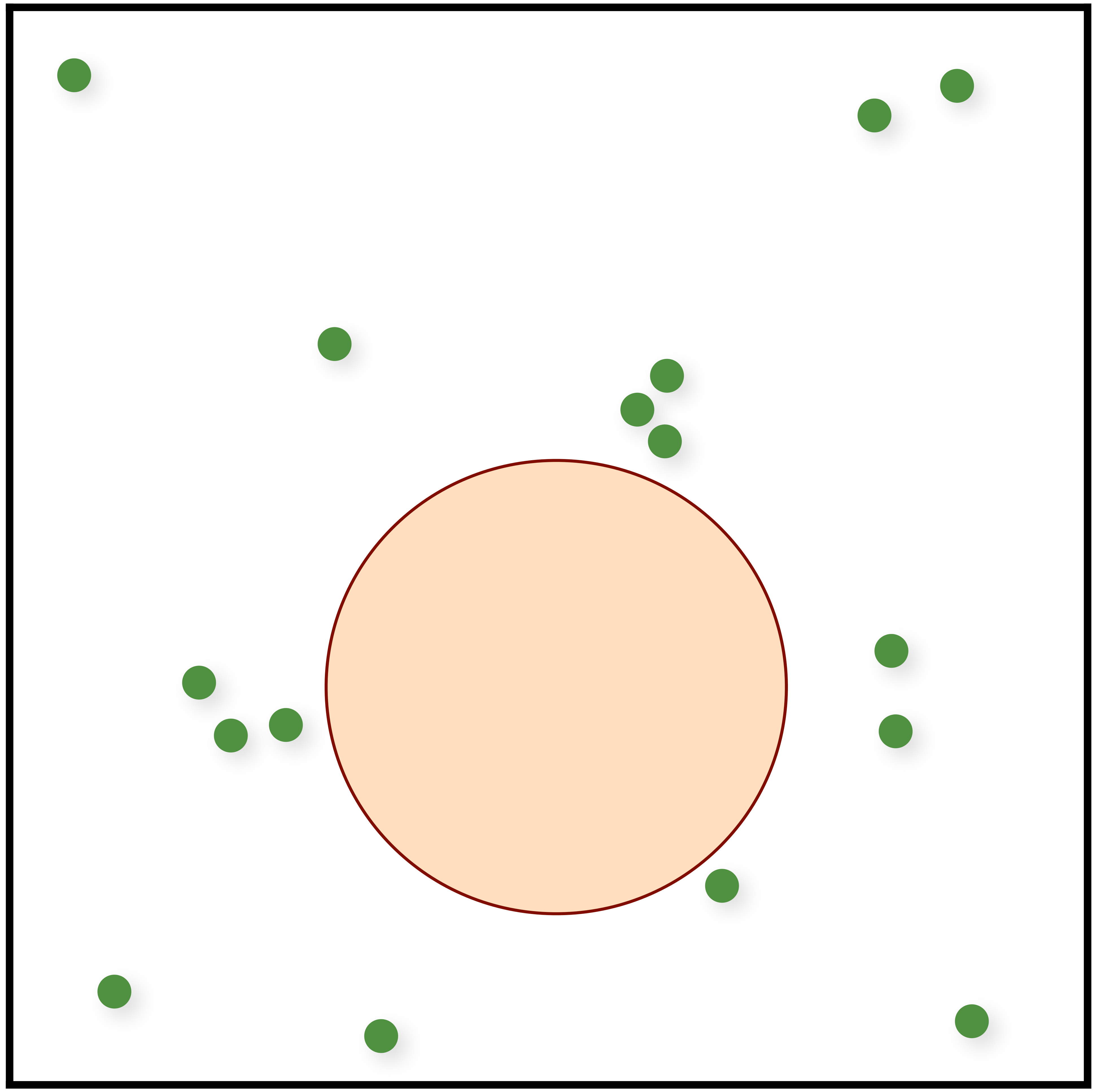

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✘ Big gaps

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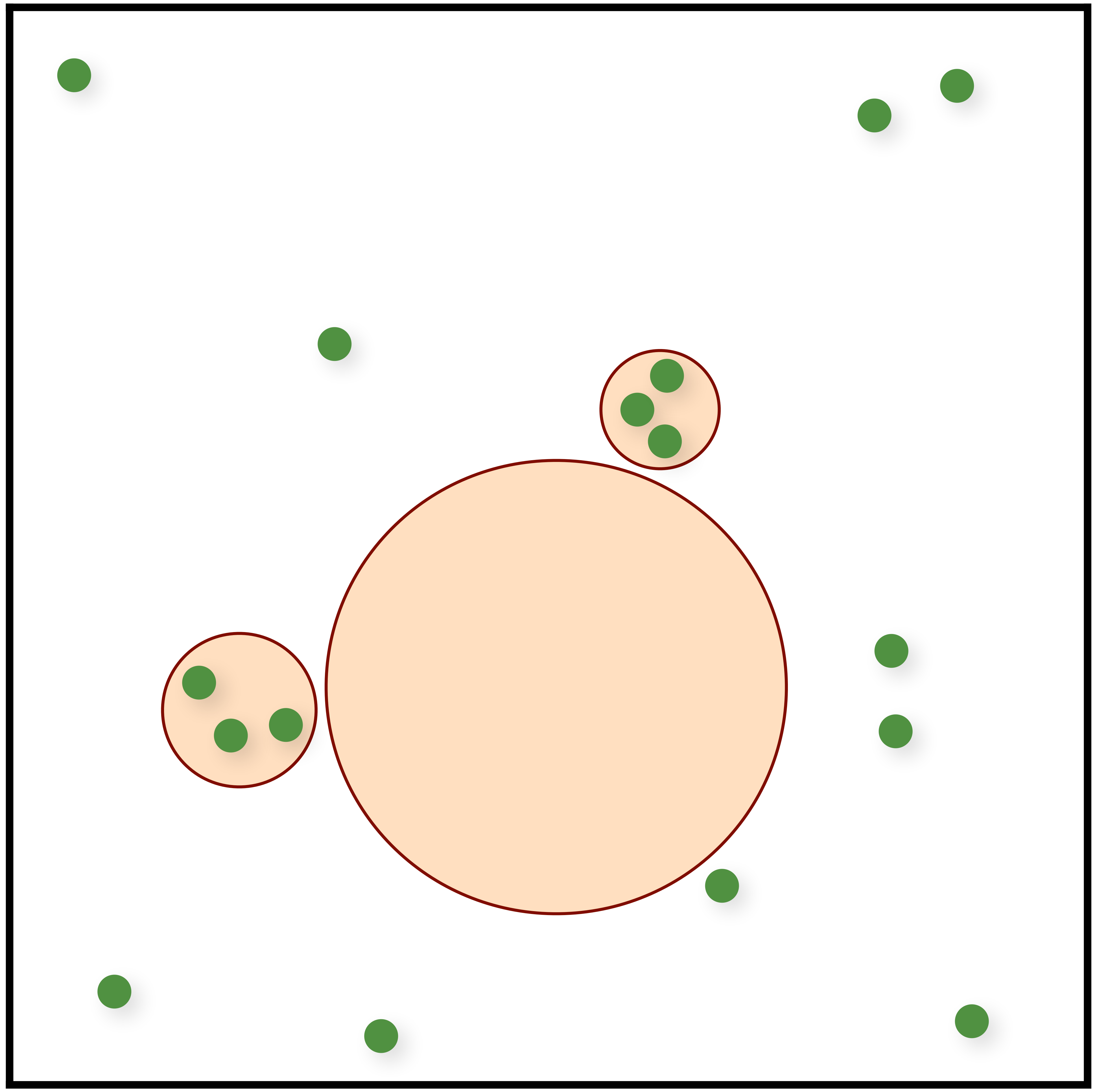

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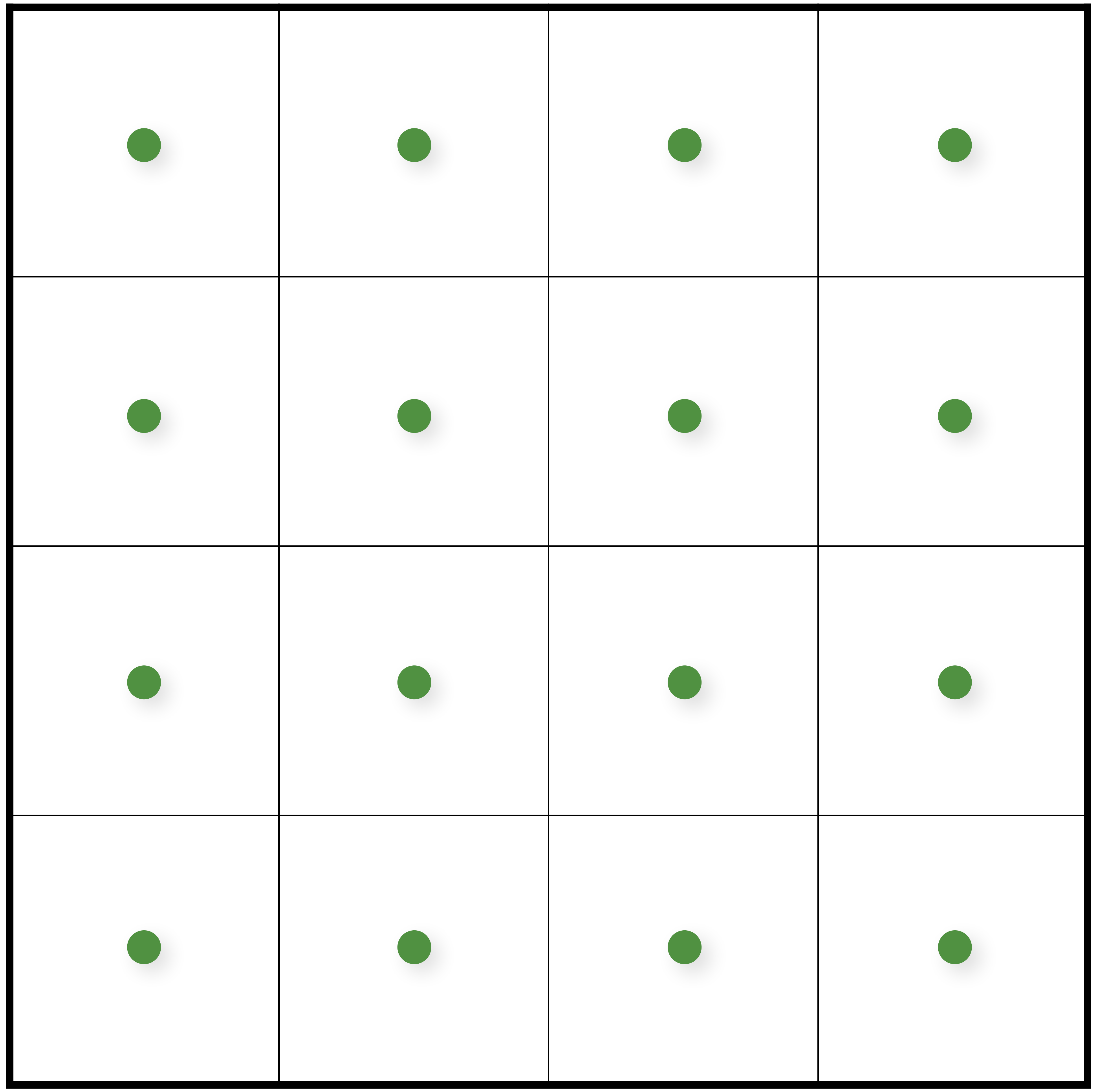
✔ Trivially progressive and memory-less

✘ Big gaps

✘ Clumping

Regular Sampling

```
for (uint i = 0; i < numX; i++)
    for (uint j = 0; j < numY; j++)
    {
        samples(i,j).x = (i + 0.5)/numX;
        samples(i,j).y = (j + 0.5)/numY;
    }
```



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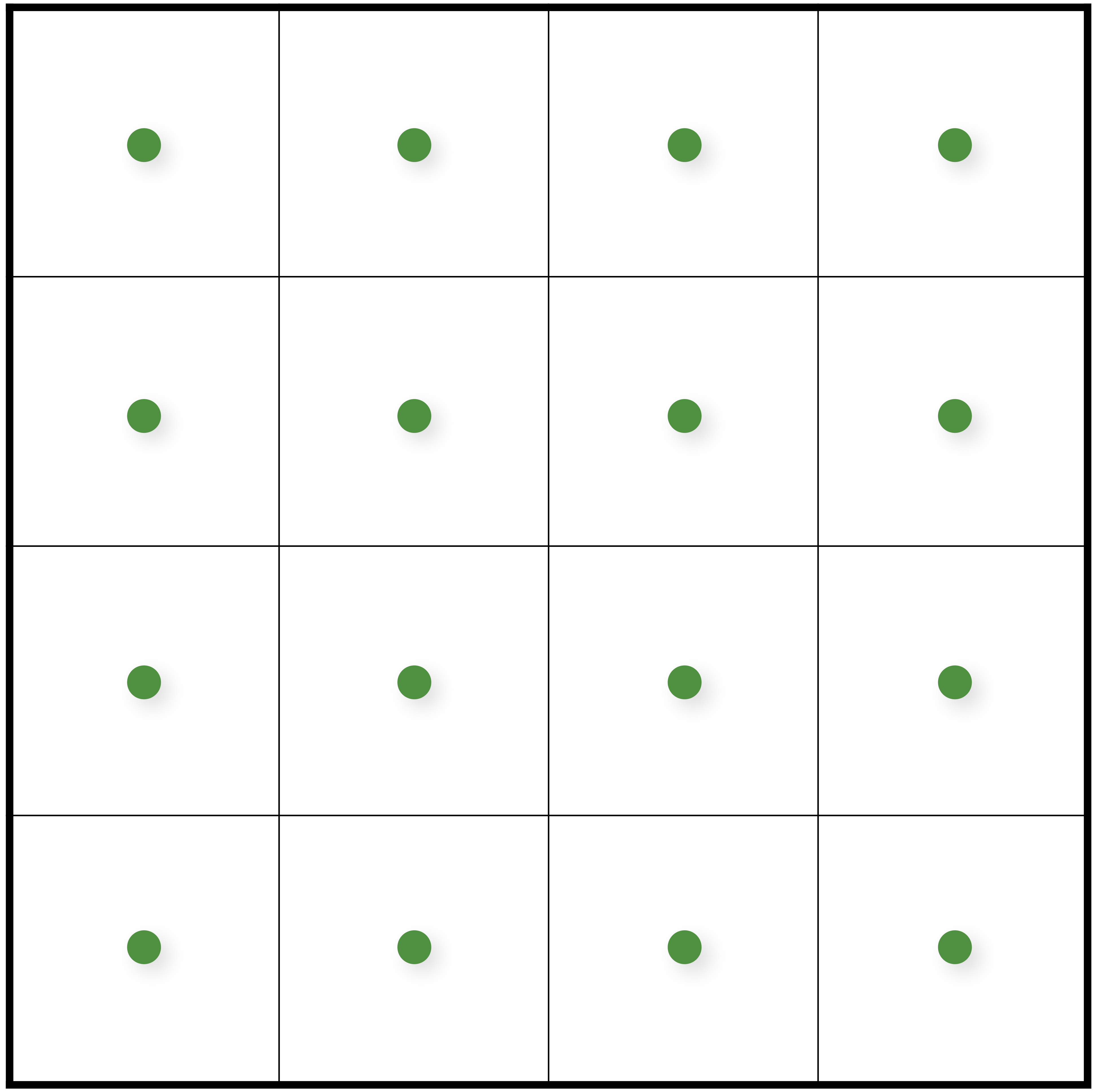

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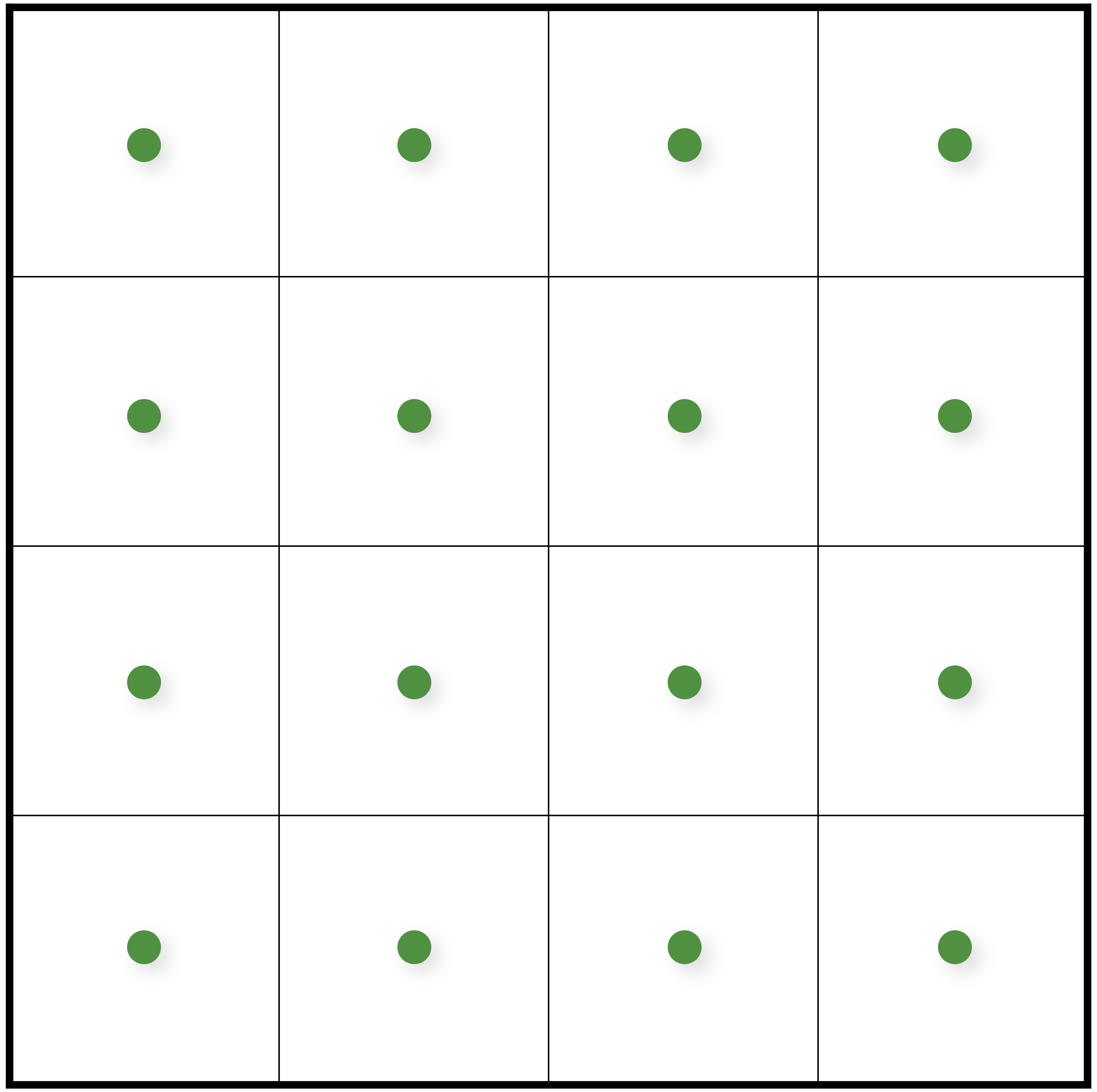

- ✔ Extends to higher dimensions, but…
- ✘ Curse of dimensionality
- ✘ Aliasing

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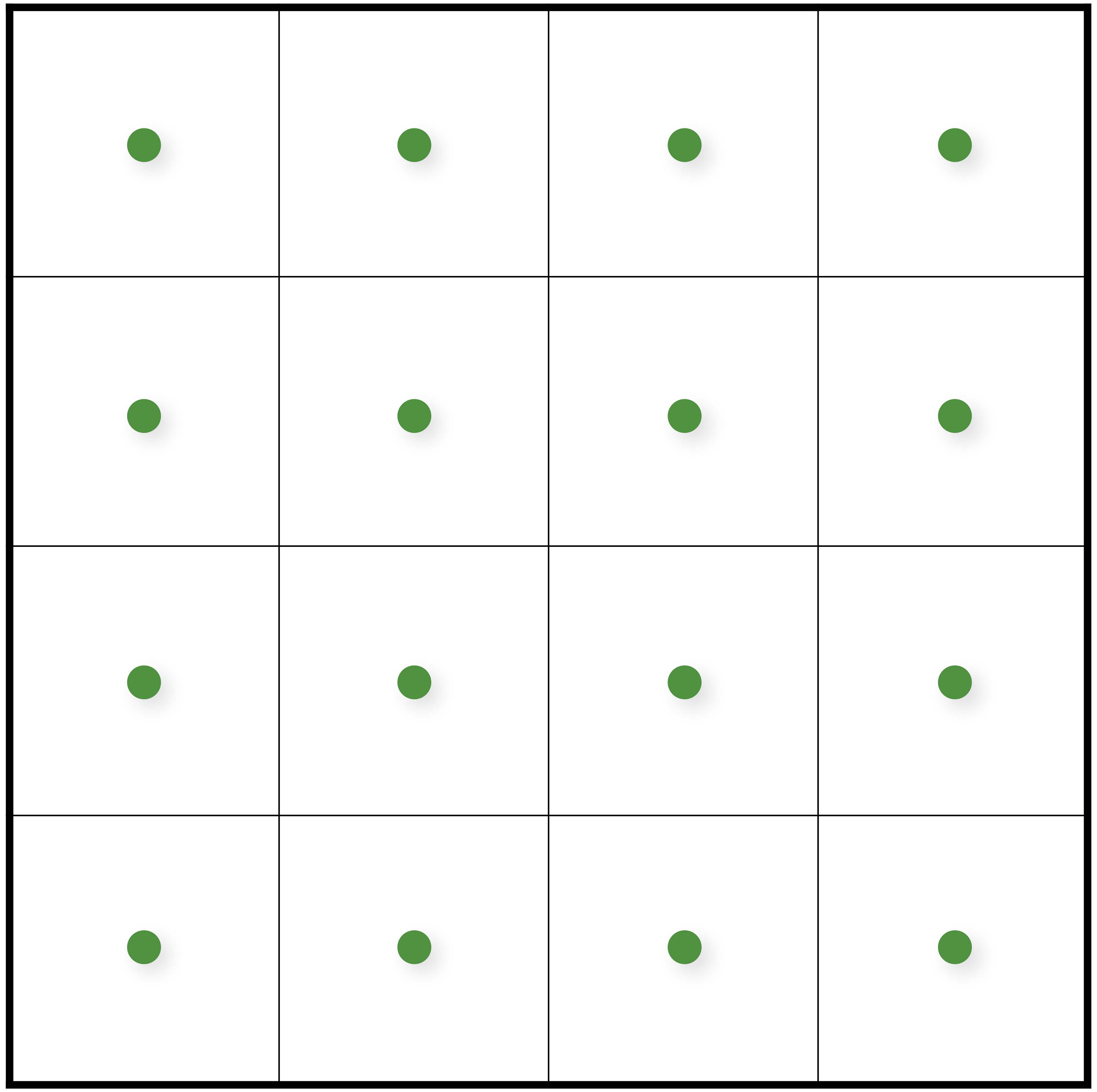

Jittered/Stratified Sampling

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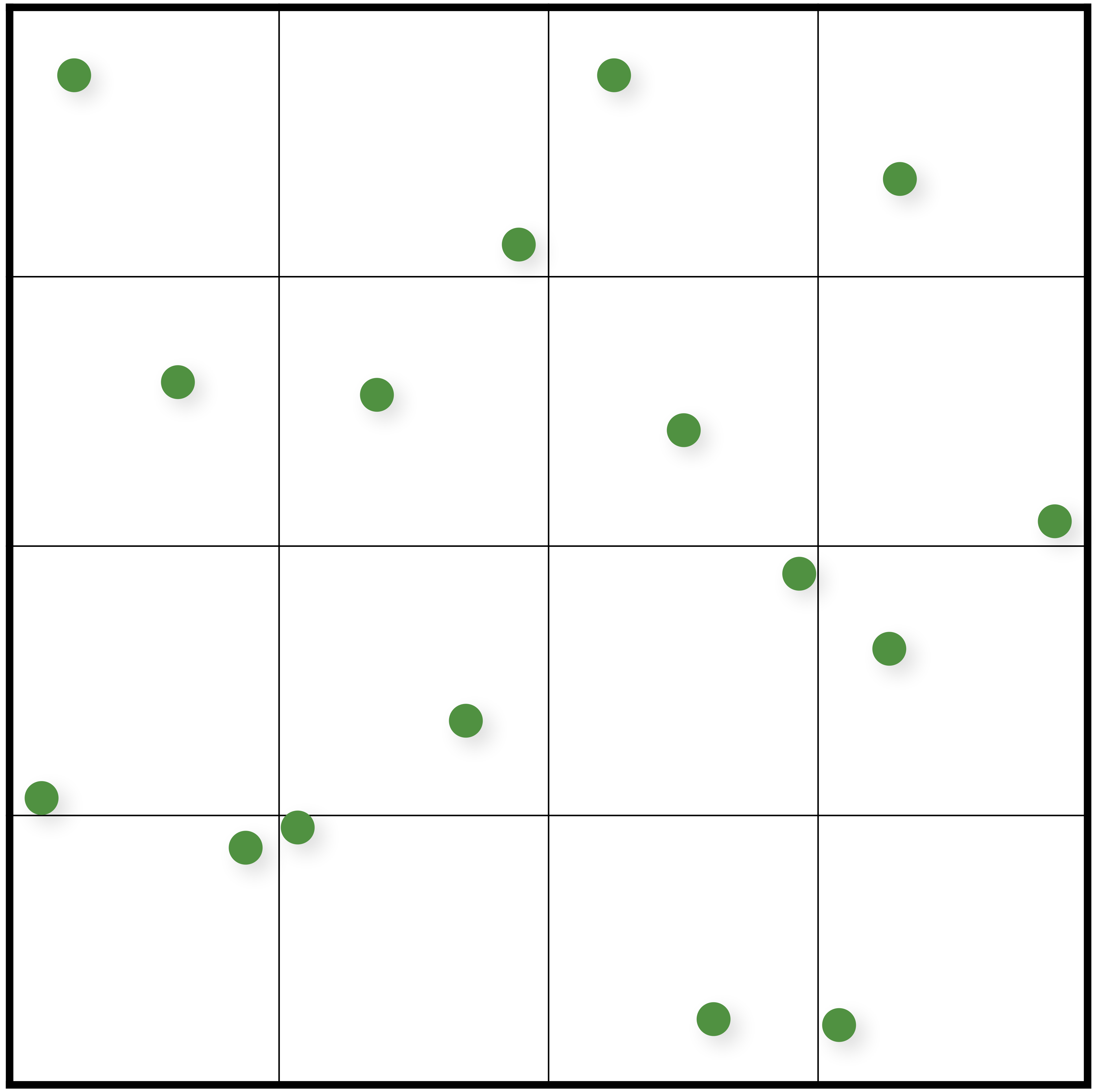

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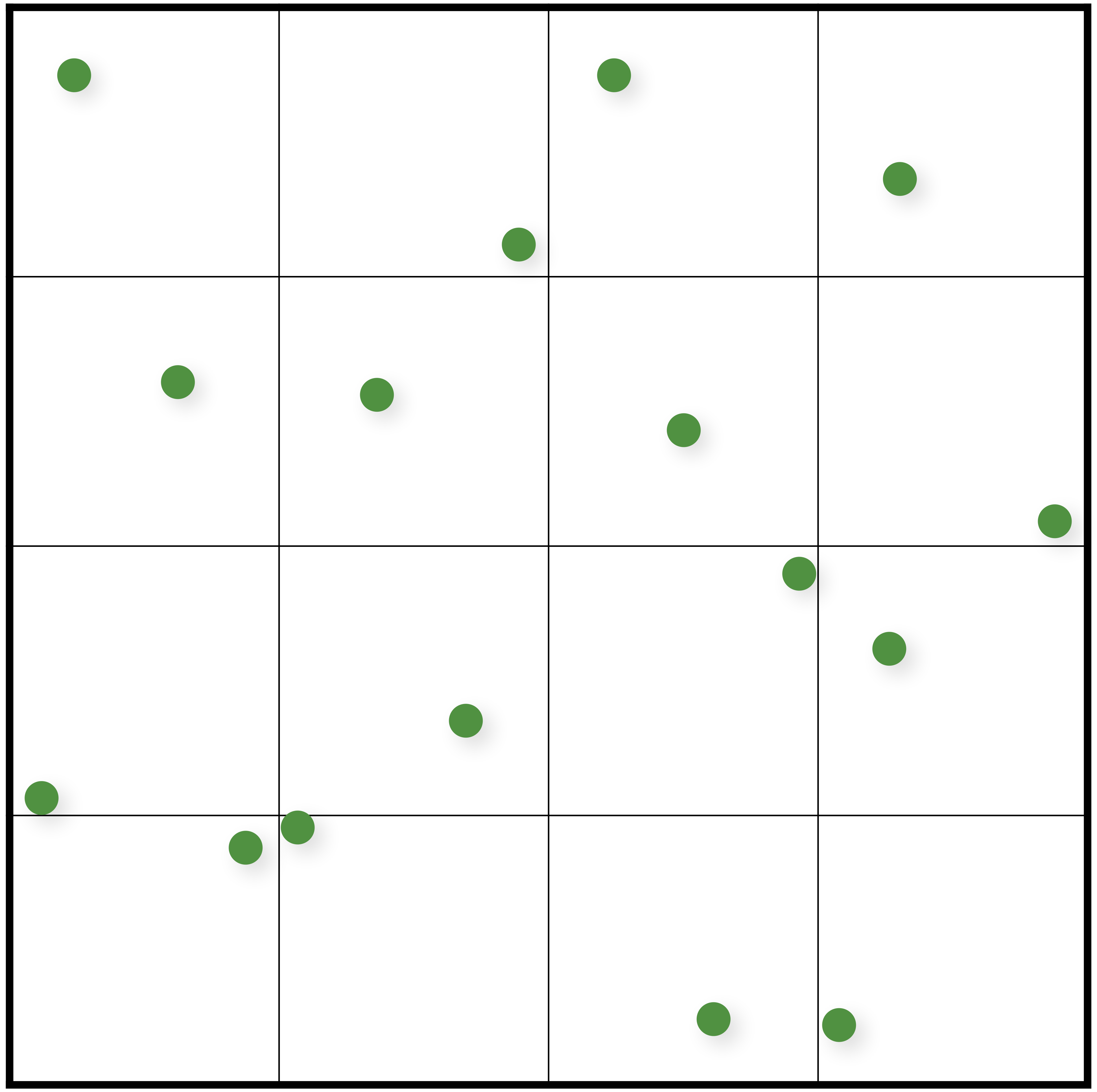

✔ Provably cannot increase variance

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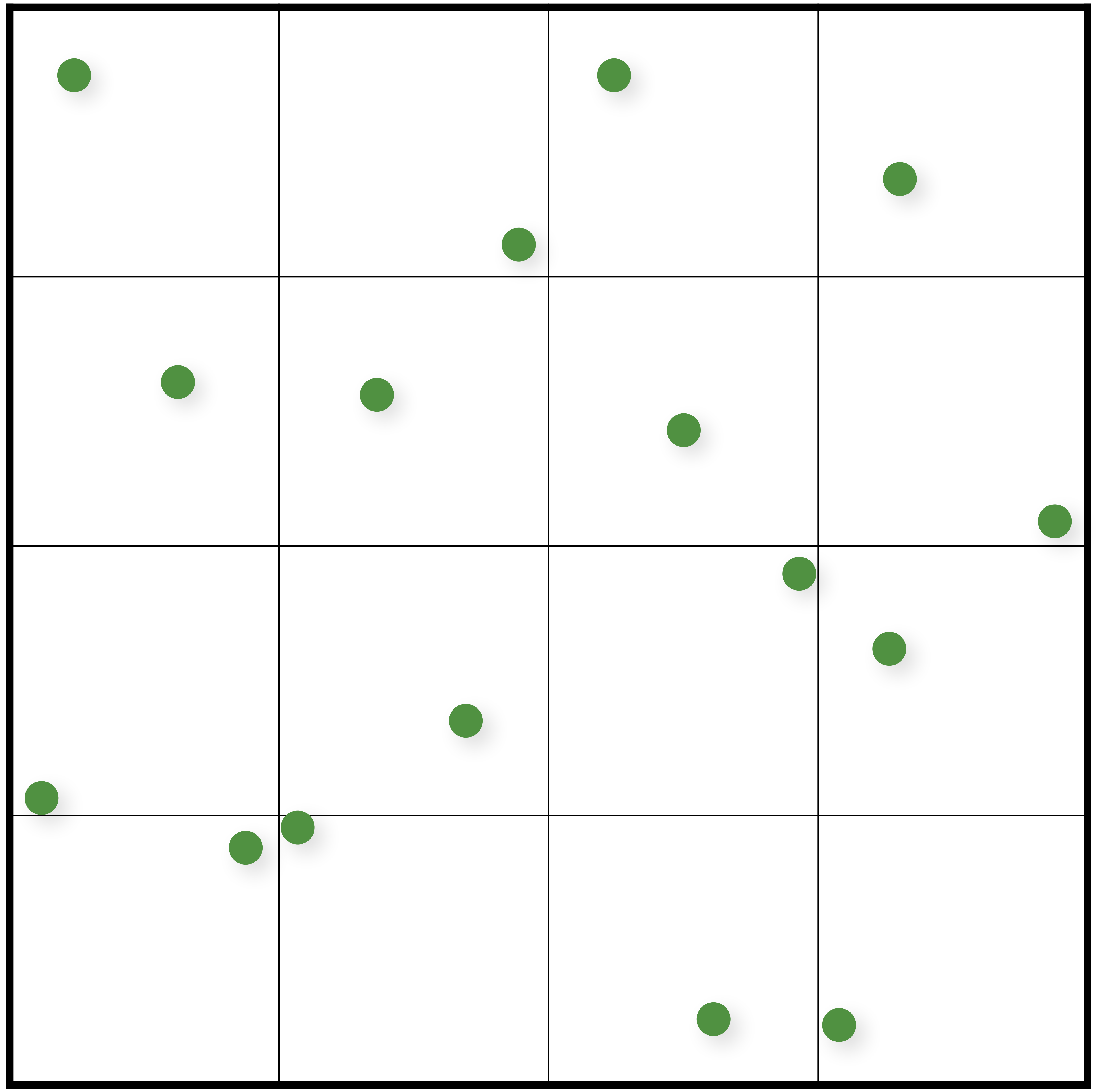

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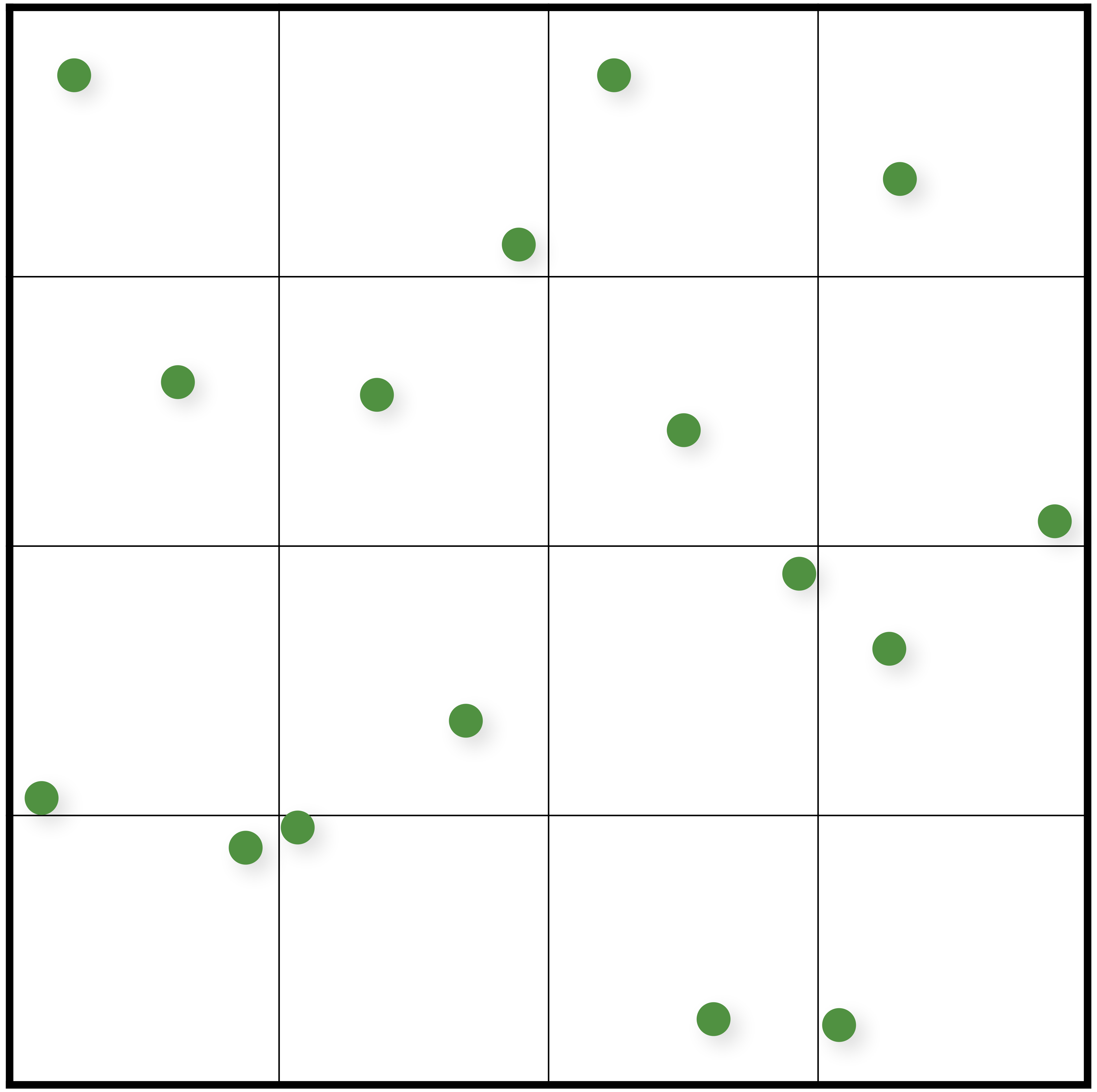

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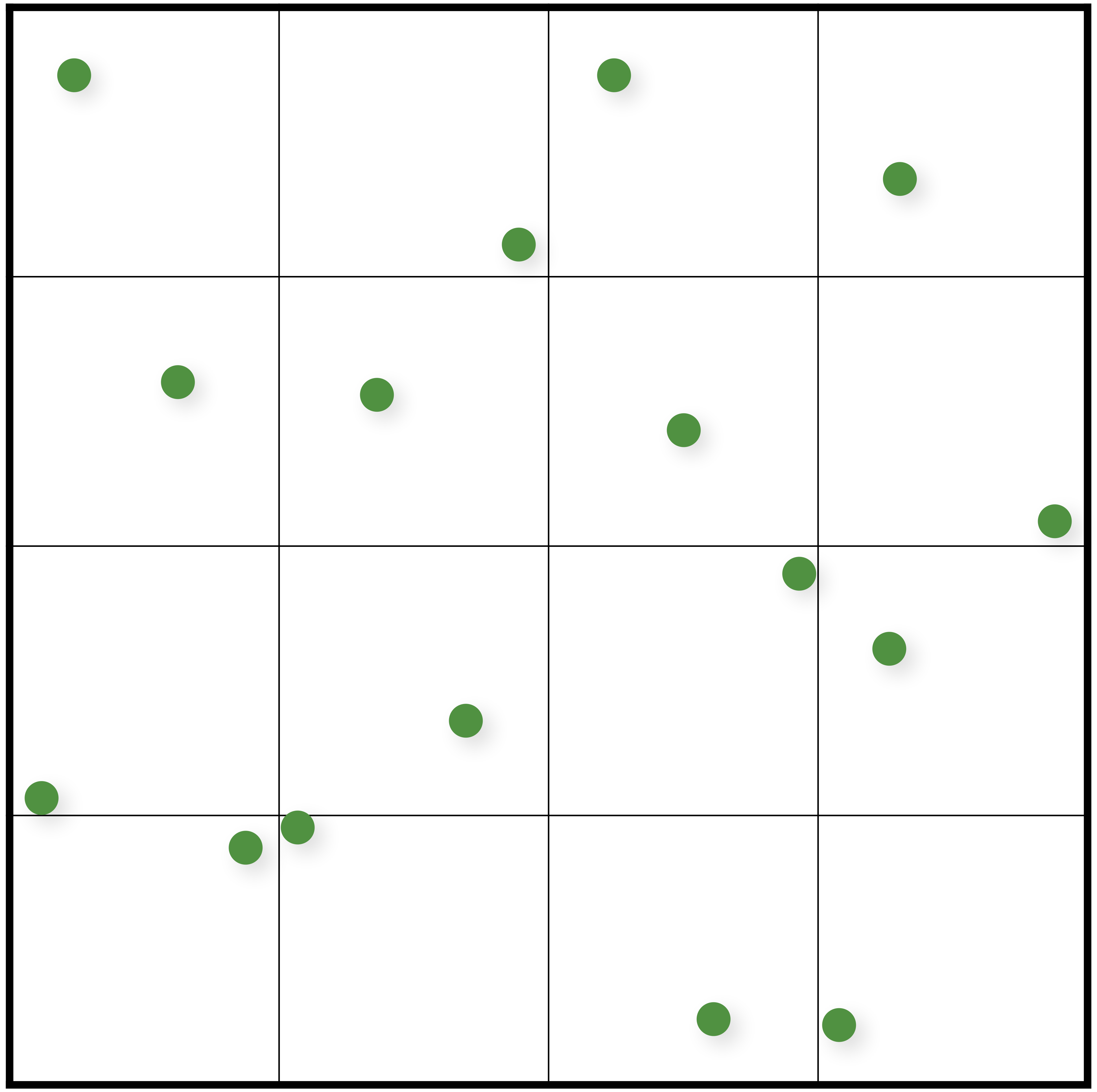
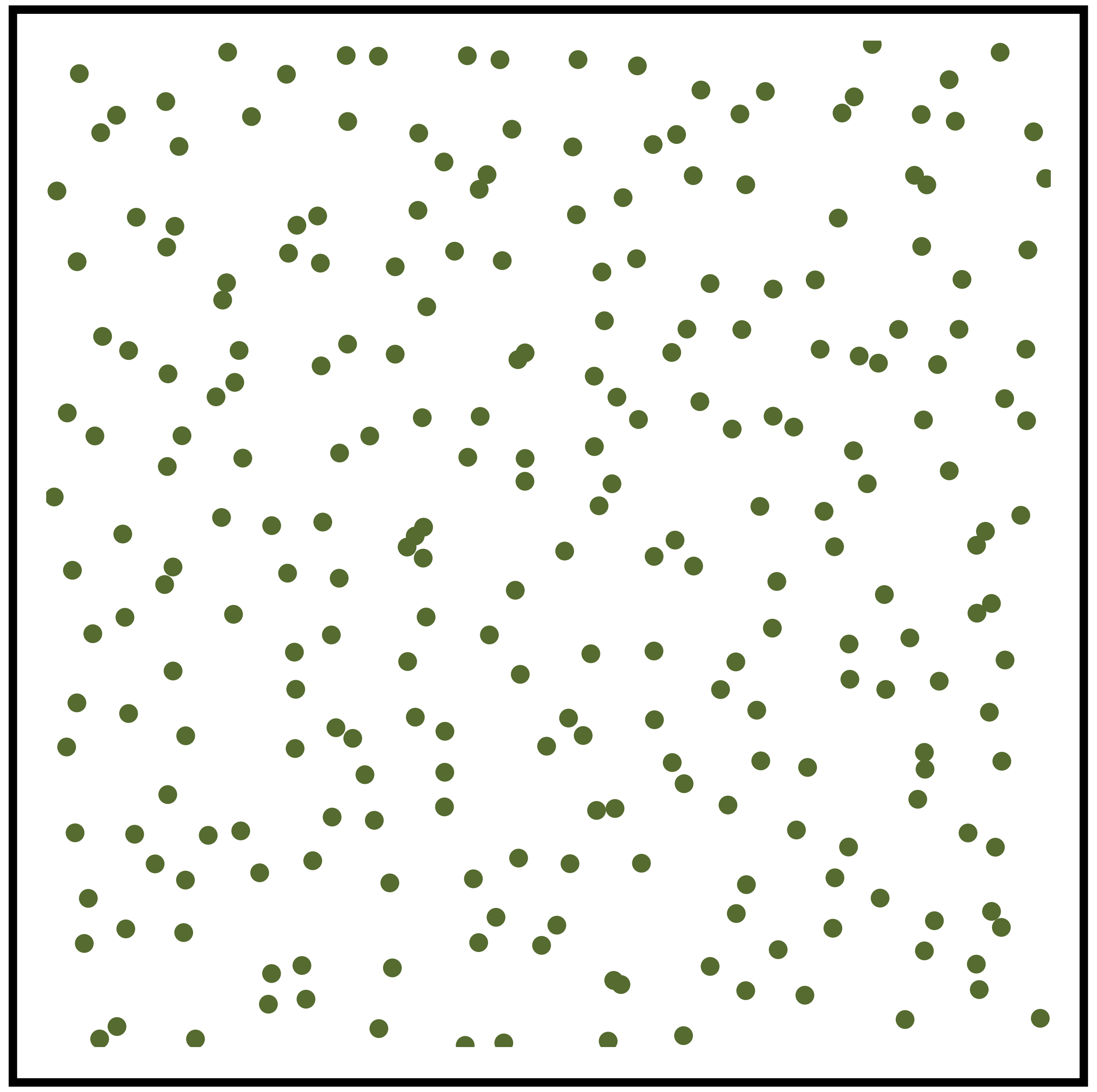


- ✔ Provably cannot increase variance
- ✔ Extends to higher dimensions, but…
- ✘ Curse of dimensionality
- ✘ Not progressive

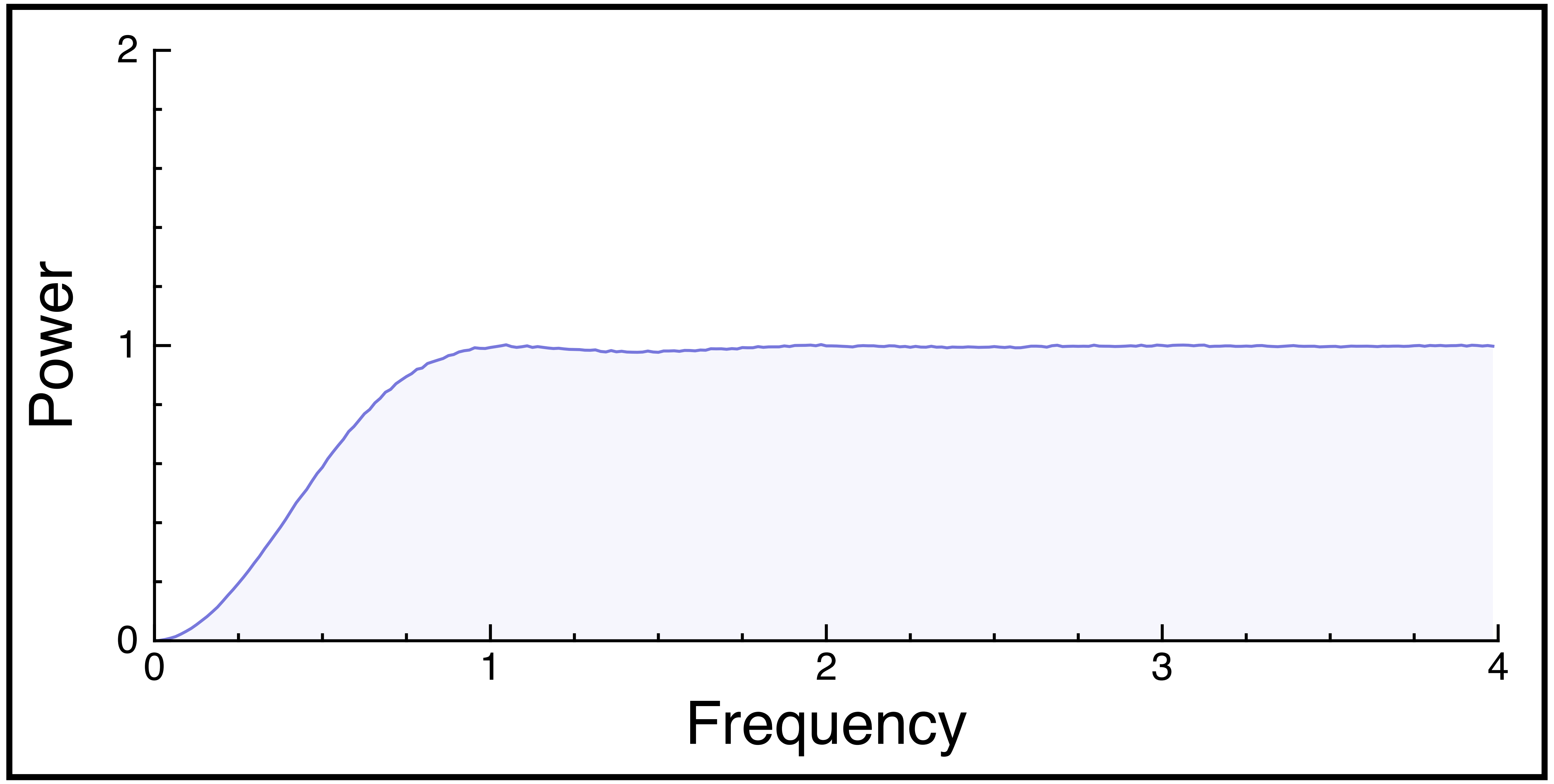
Jittered Sampling

Samples



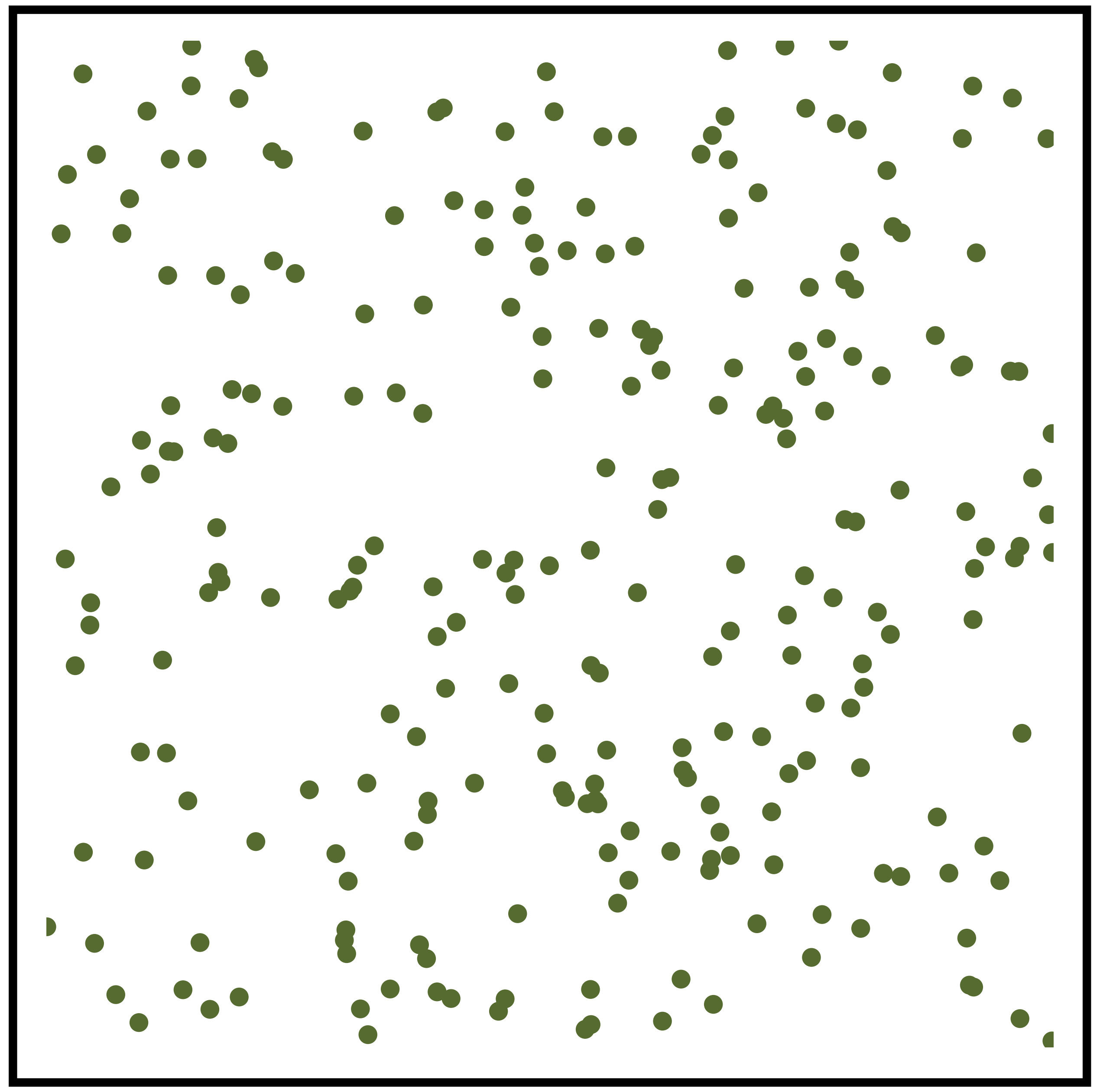
Expected power spectrum

Radial mean



Independent Random Sampling

Samples



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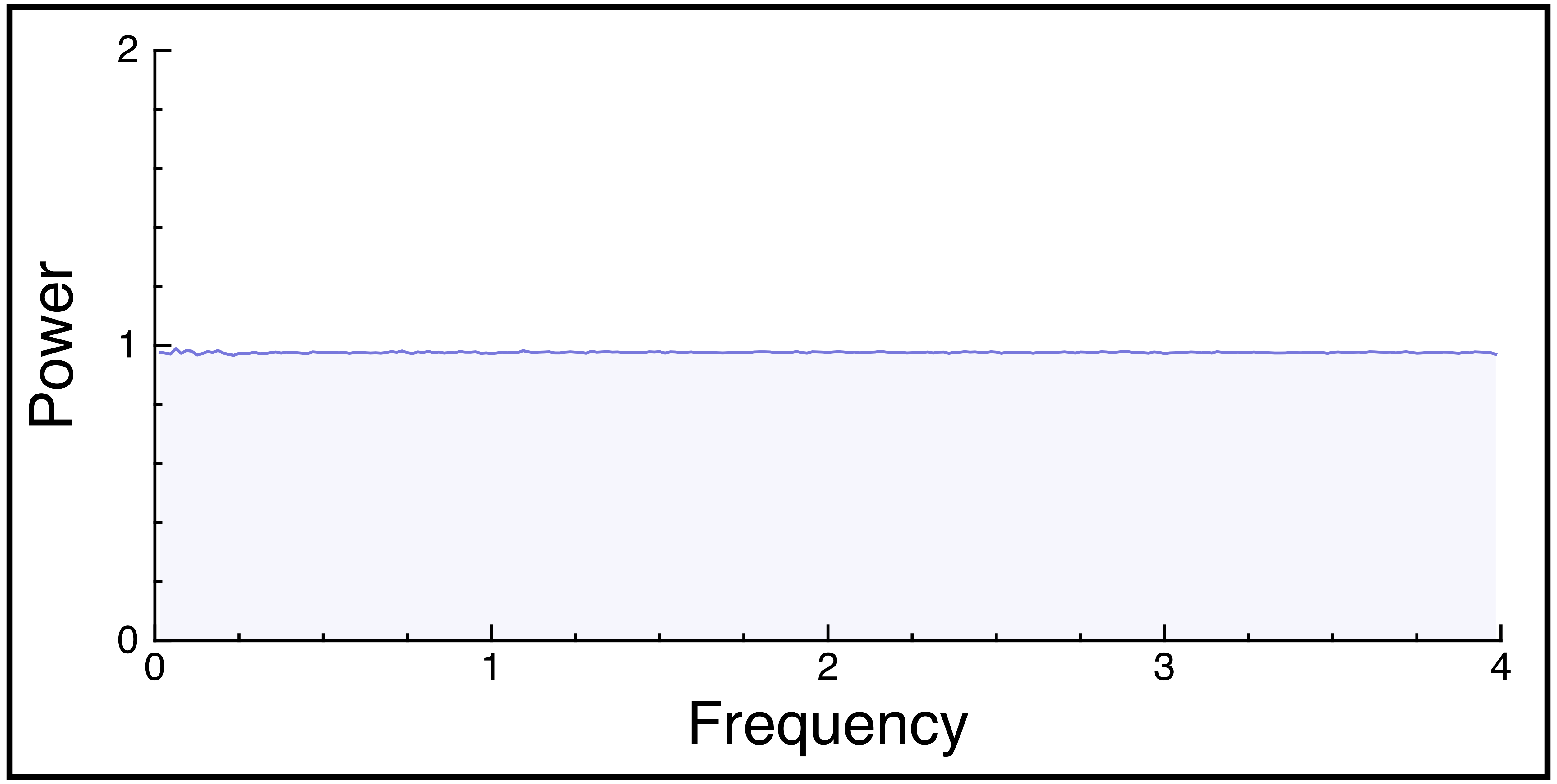

Monte Carlo (16 random samples)

Monte Carlo (16 jittered samples)

Stratifying in Higher Dimensions

Stratification requires O(N^d) samples

- e.g. pixel (2D) + lens (2D) + time (1D) = 5D

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  - splitting 2 times in 5D = 2^5 = 32 samples
  - splitting 3 times in 5D = 3^5 = 243 samples!

Inconvenient for large d

- cannot select sample count with fine granularity

Uncorrelated Jitter [Cook et al. 84]

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Compute stratified samples in sub-dimensions

Uncorrelated Jitter [Cook et al. 84]

Compute stratified samples in sub-dimensions

- 2D jittered (x,y) for pixel

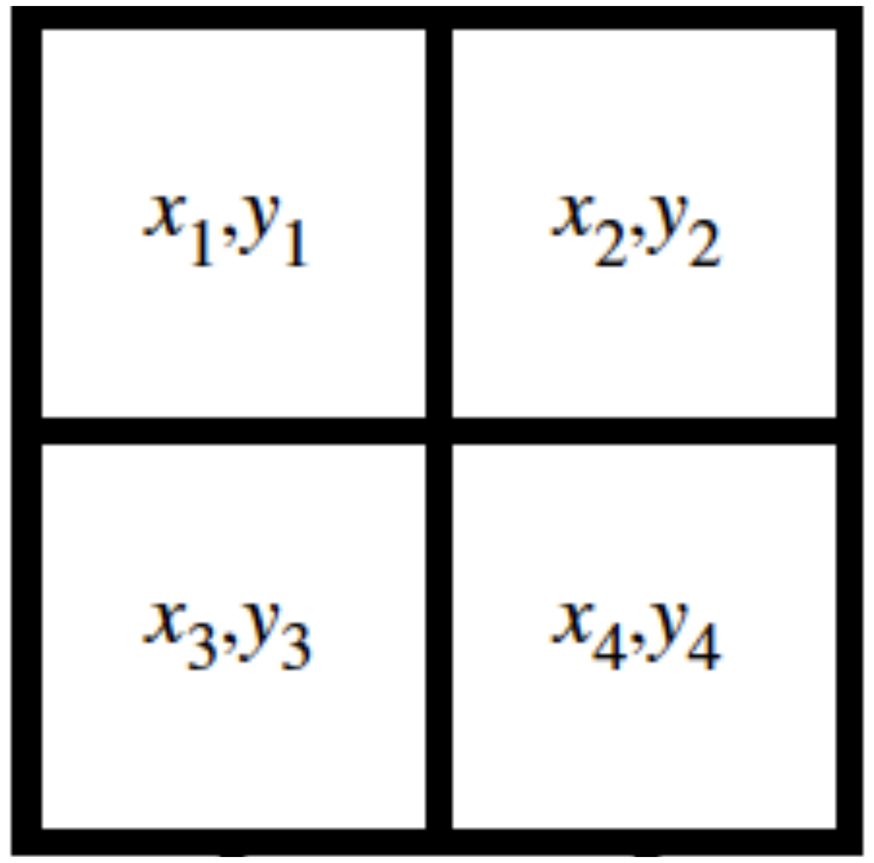
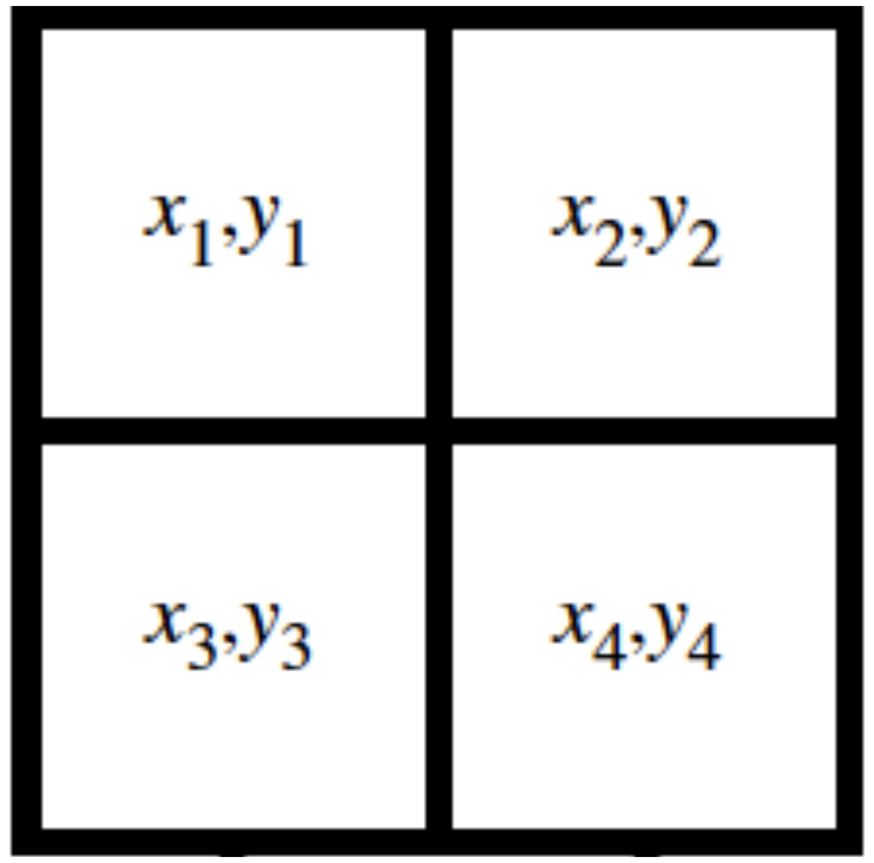


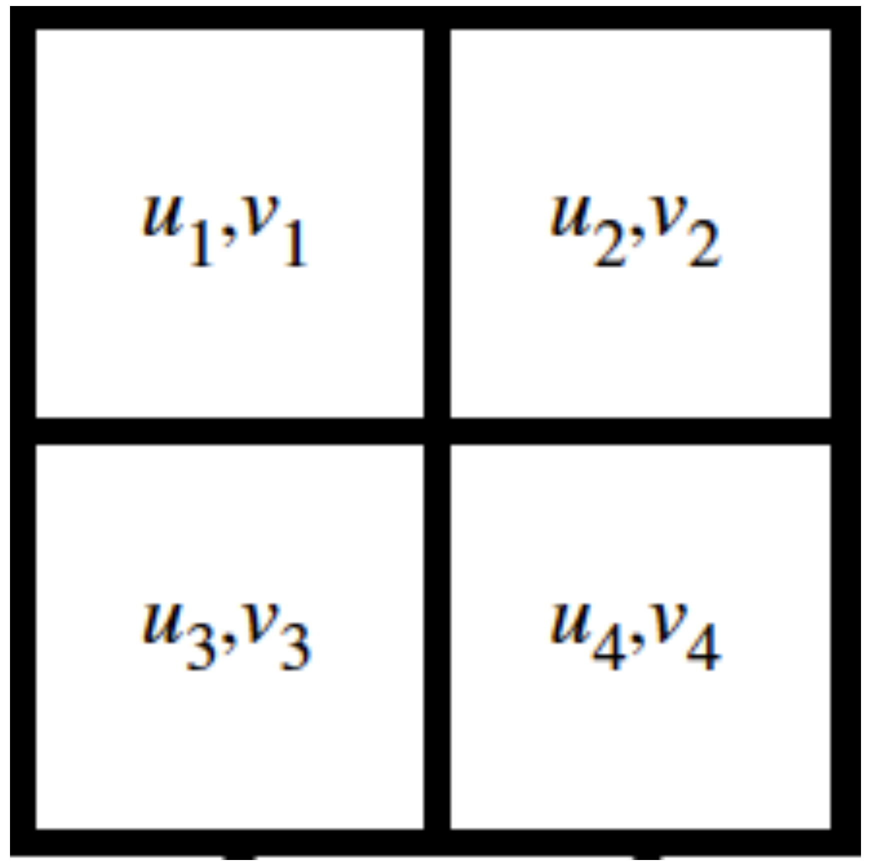
Image source: PBRTe2 [Pharr & Humphreys 2010]

Uncorrelated Jitter [Cook et al. 84]

Compute stratified samples in sub-dimensions

- 2D jittered (x,y) for pixel
- 2D jittered (u,v) for lens





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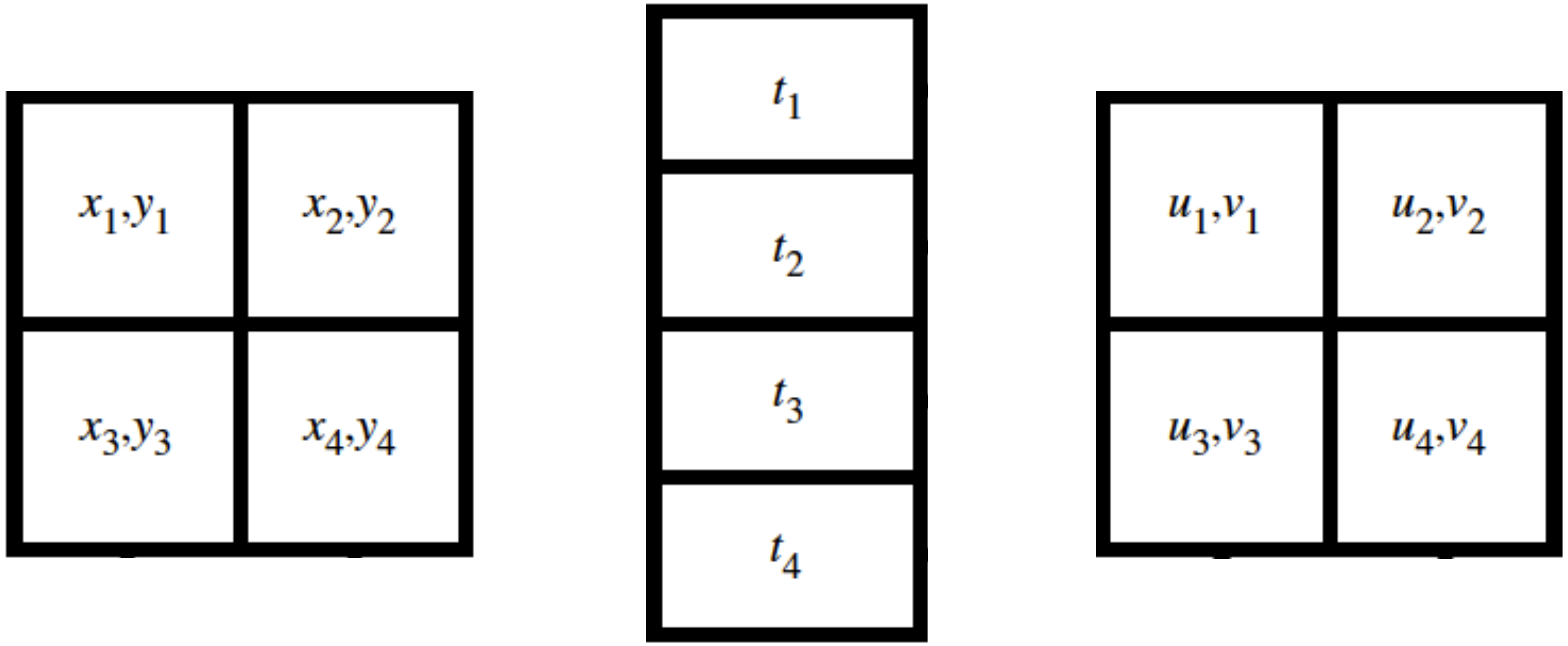

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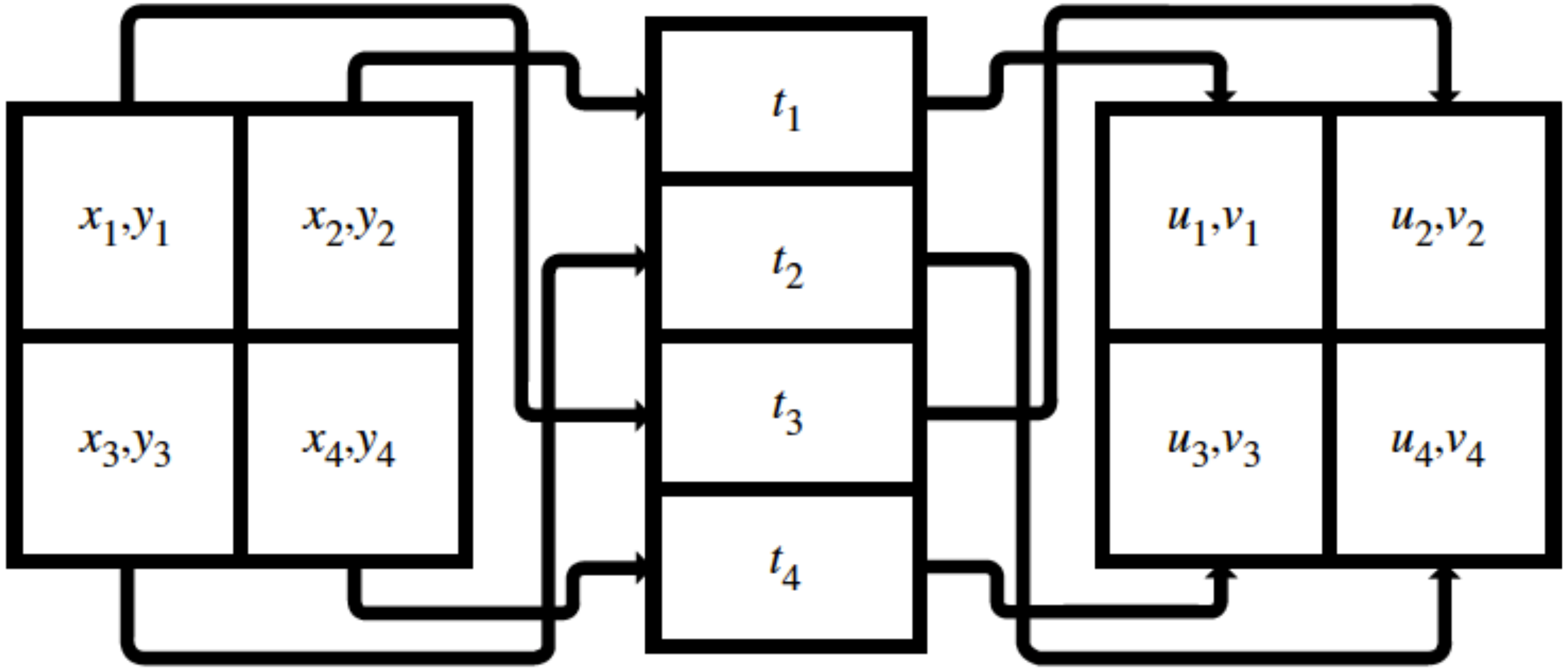
Compute stratified samples in sub-dimensions

- 2D jittered (x,y) for pixel
- 2D jittered (u,v) for lens
- 1D jittered (t) for time
- combine dimensions in random order



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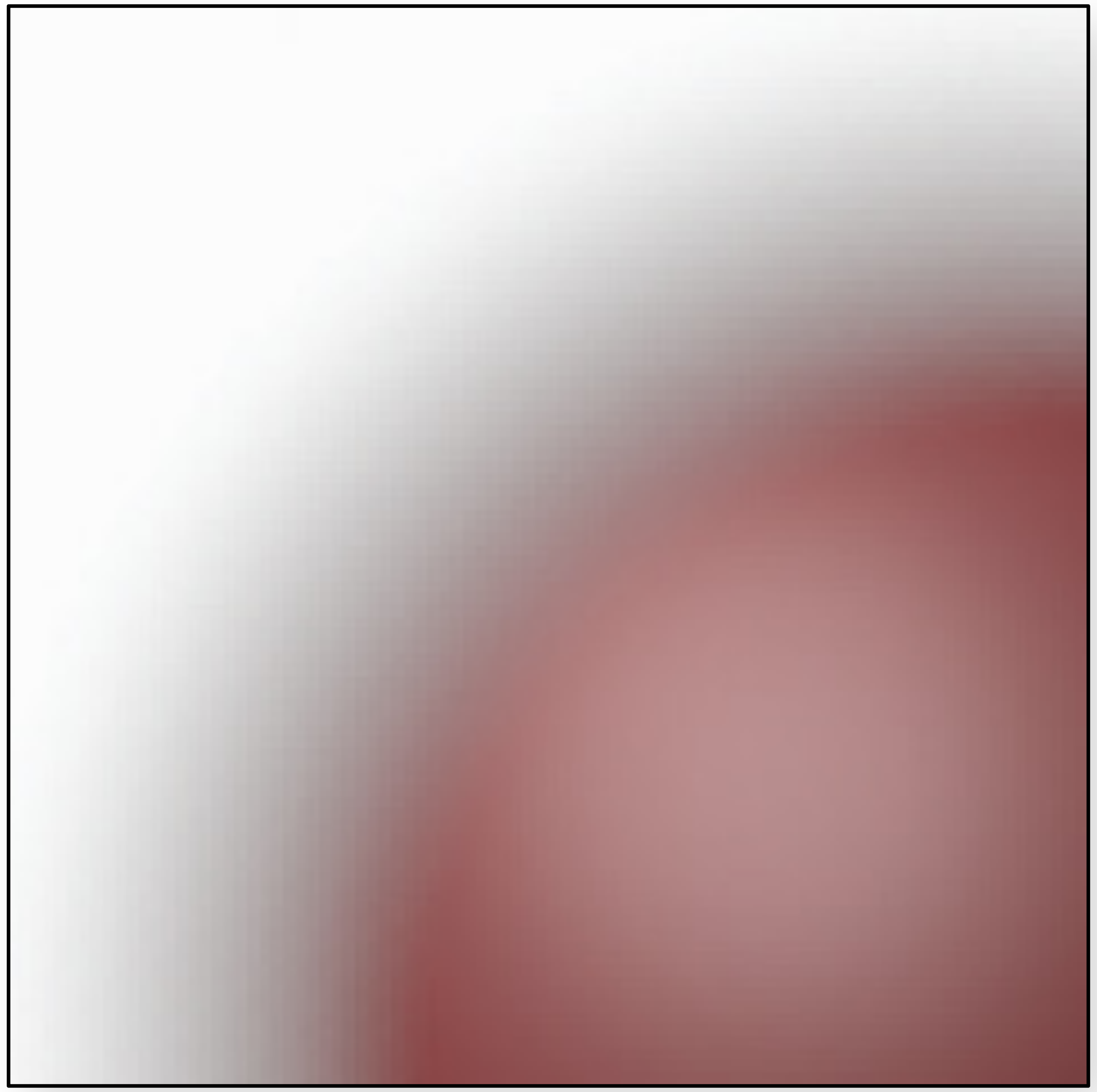
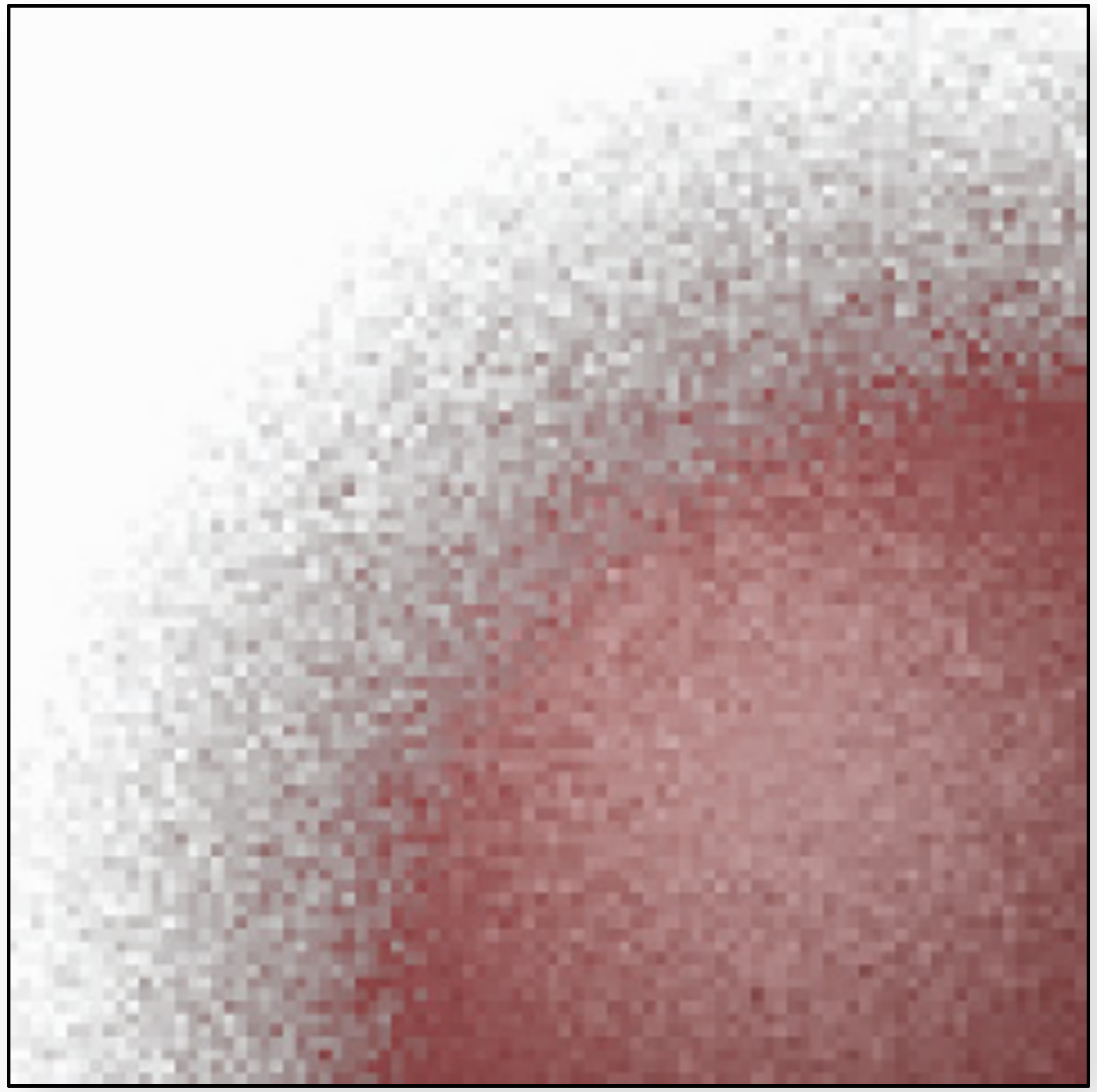
Depth of Field (4D)

Reference



Random Sampling



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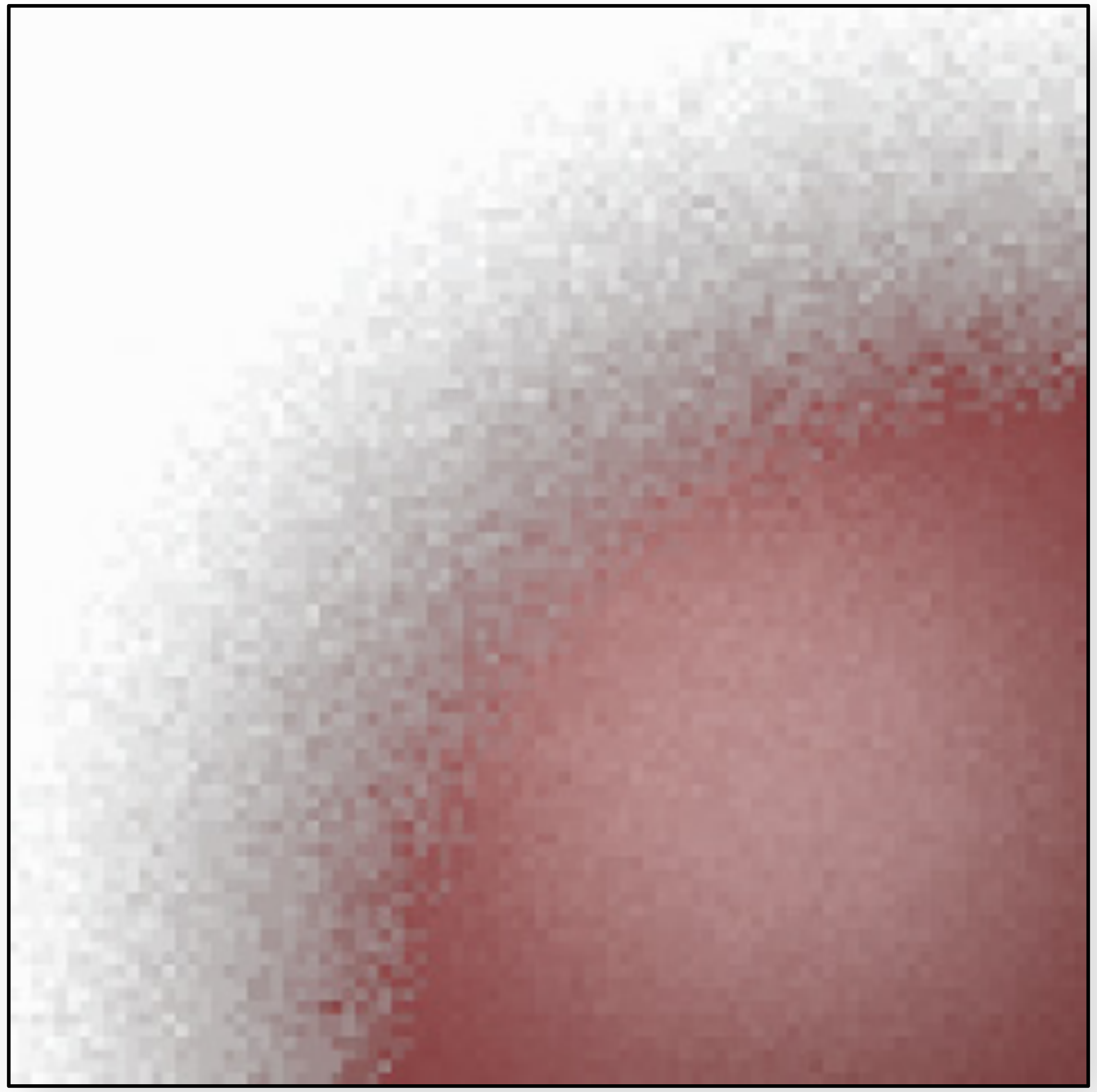

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Latin Hypercube (N-Rooks) Sampling

[Shirley 91]

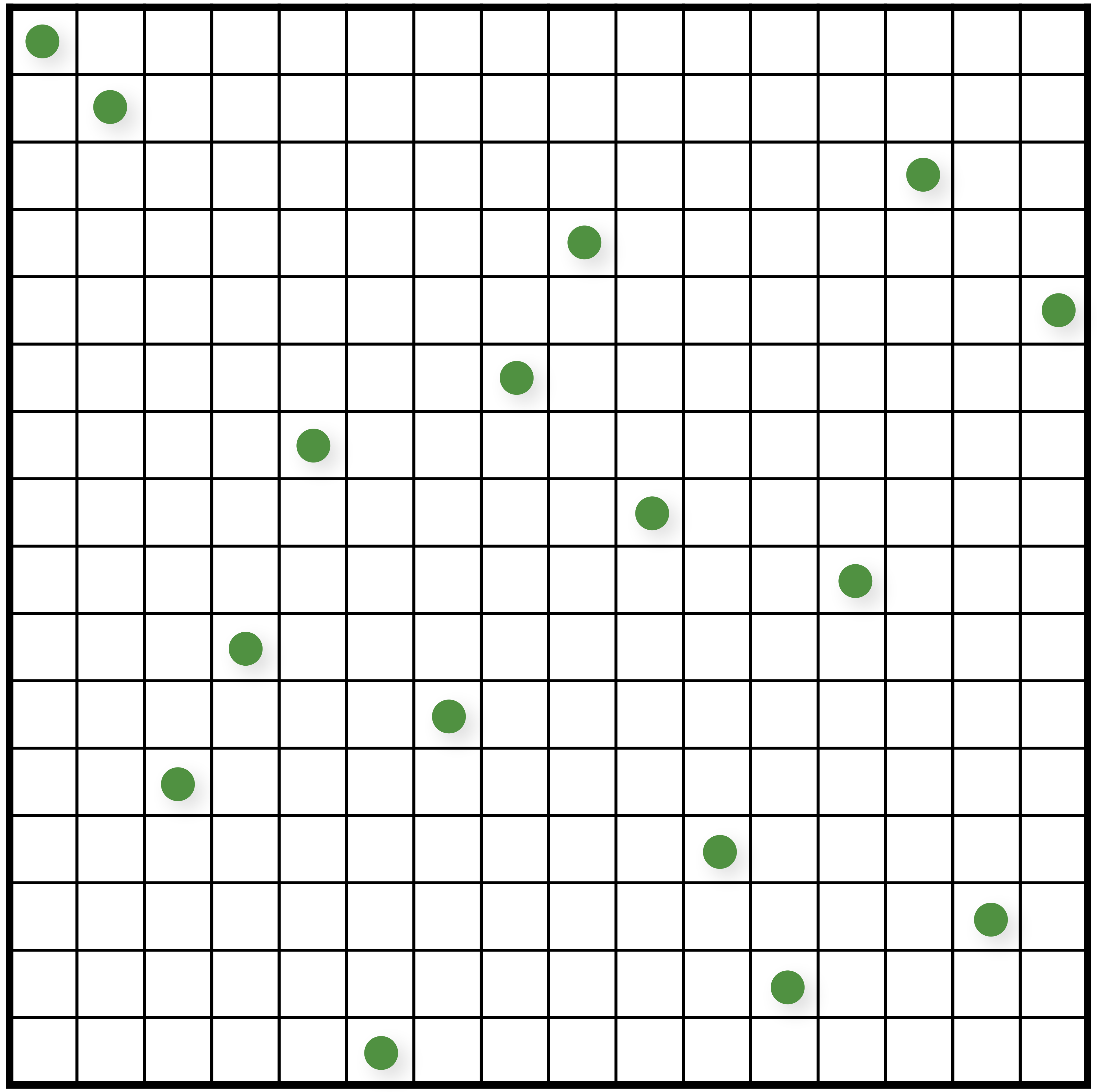
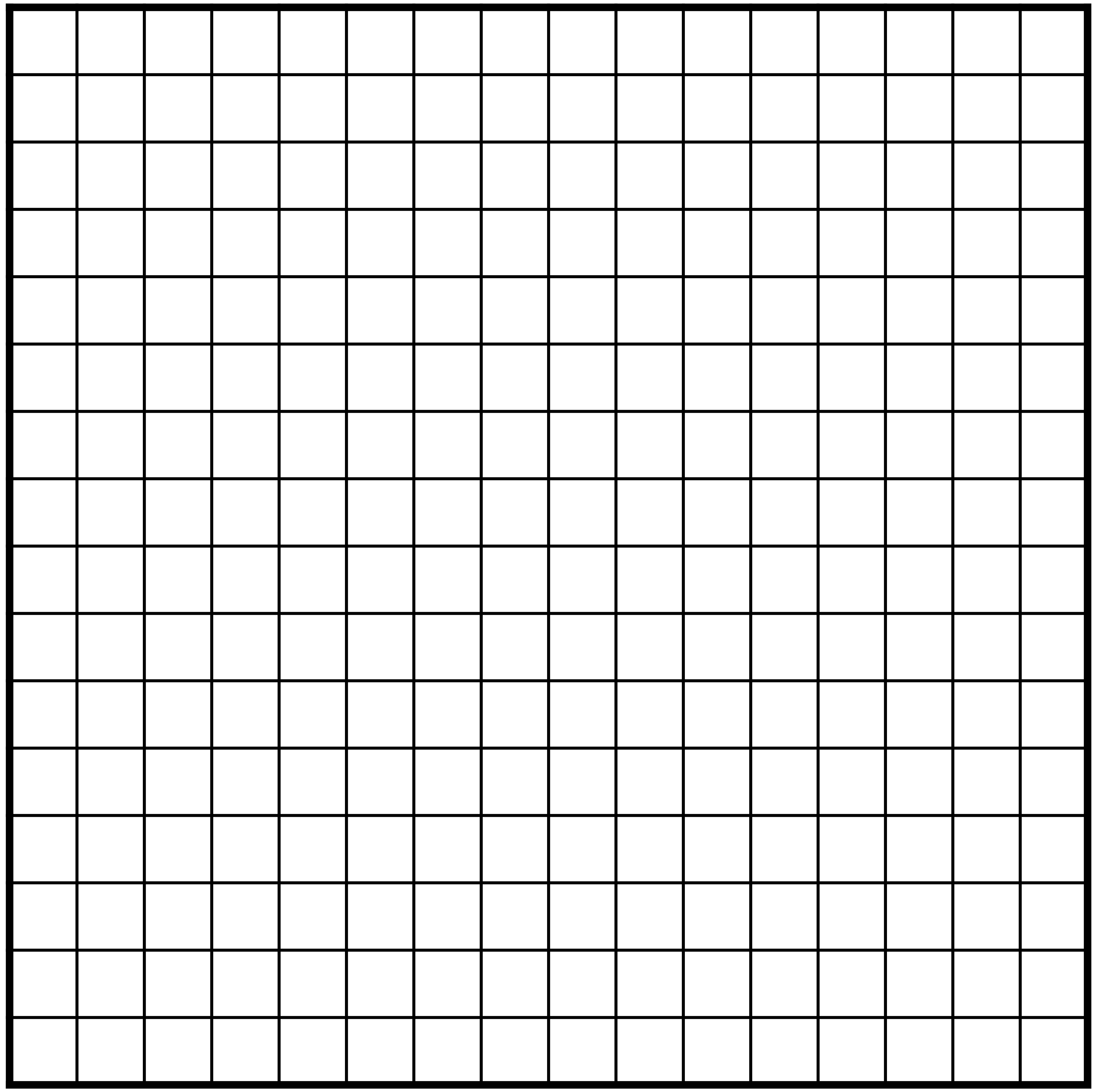


Image source: Michael Maggs, CC BY-SA 2.5

Latin Hypercube (N-Rooks) Sampling

```
// initialize the diagonal
for (uint d = 0; d < numDimensions; d++)
    for (uint i = 0; i < numS; i++)
        samples(d,i) = (i + randf())/numS;

// shuffle each dimension independently
for (uint d = 0; d < numDimensions; d++)
    shuffle(samples(d,:));
```



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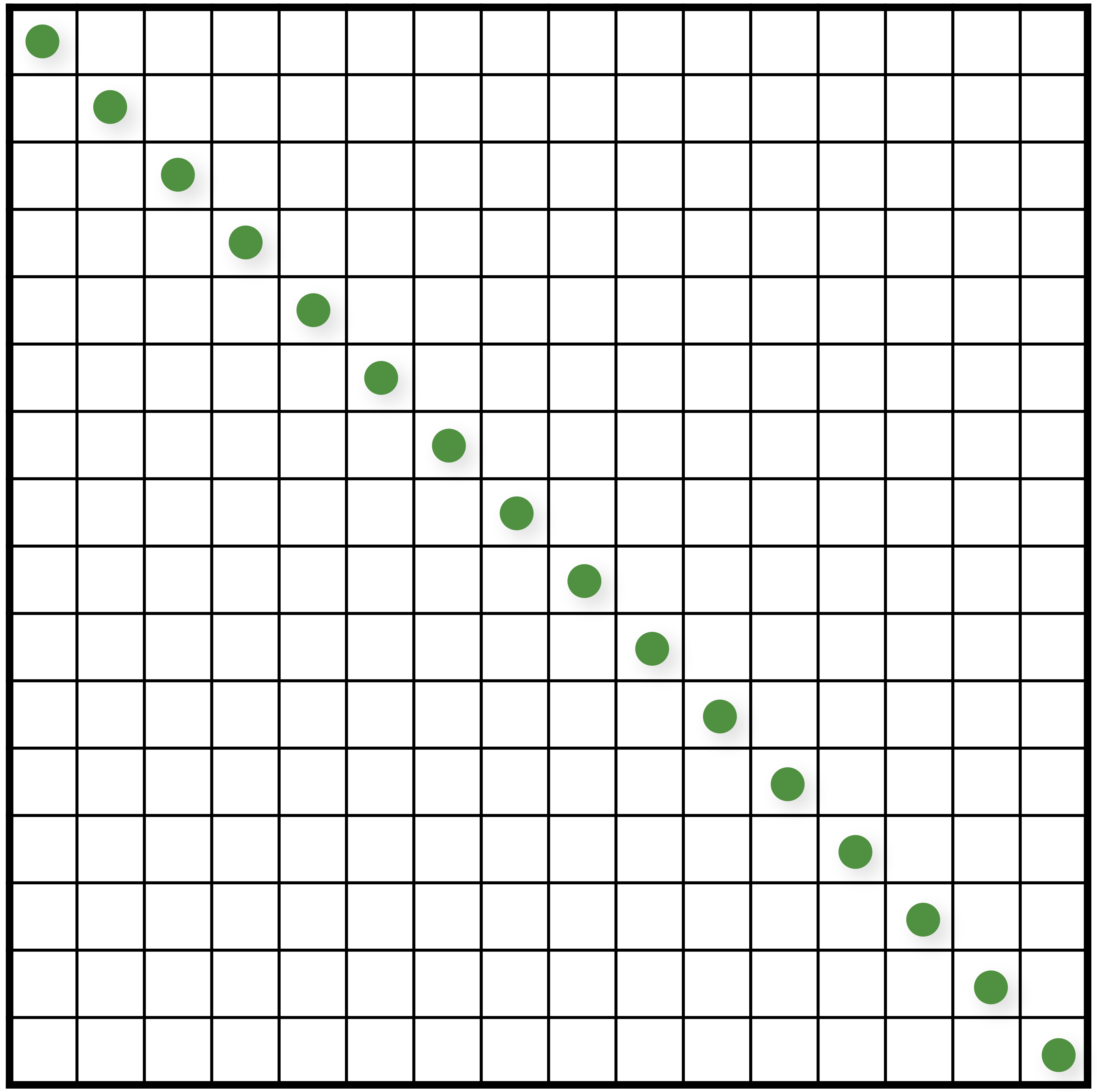

Initialize

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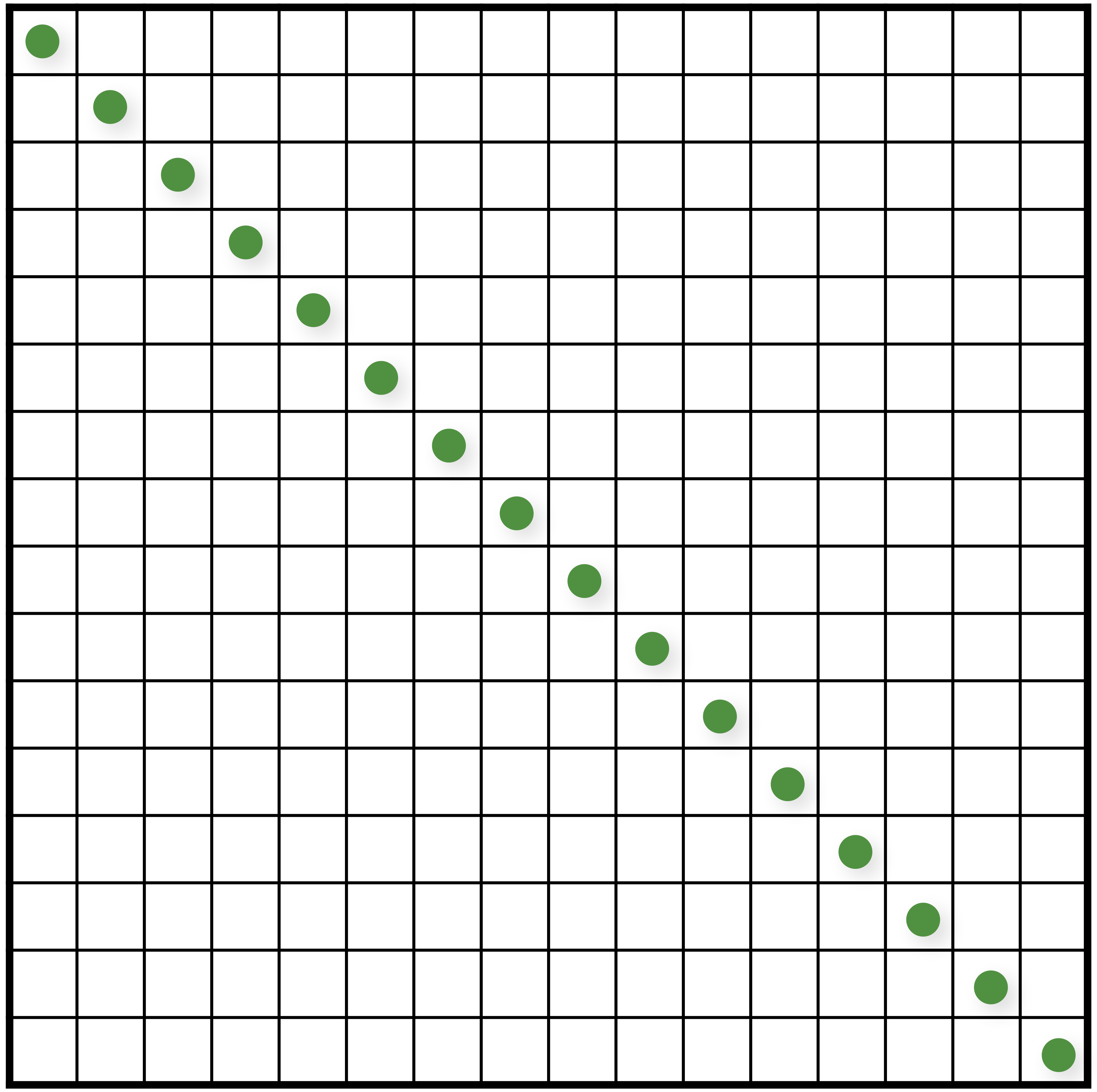

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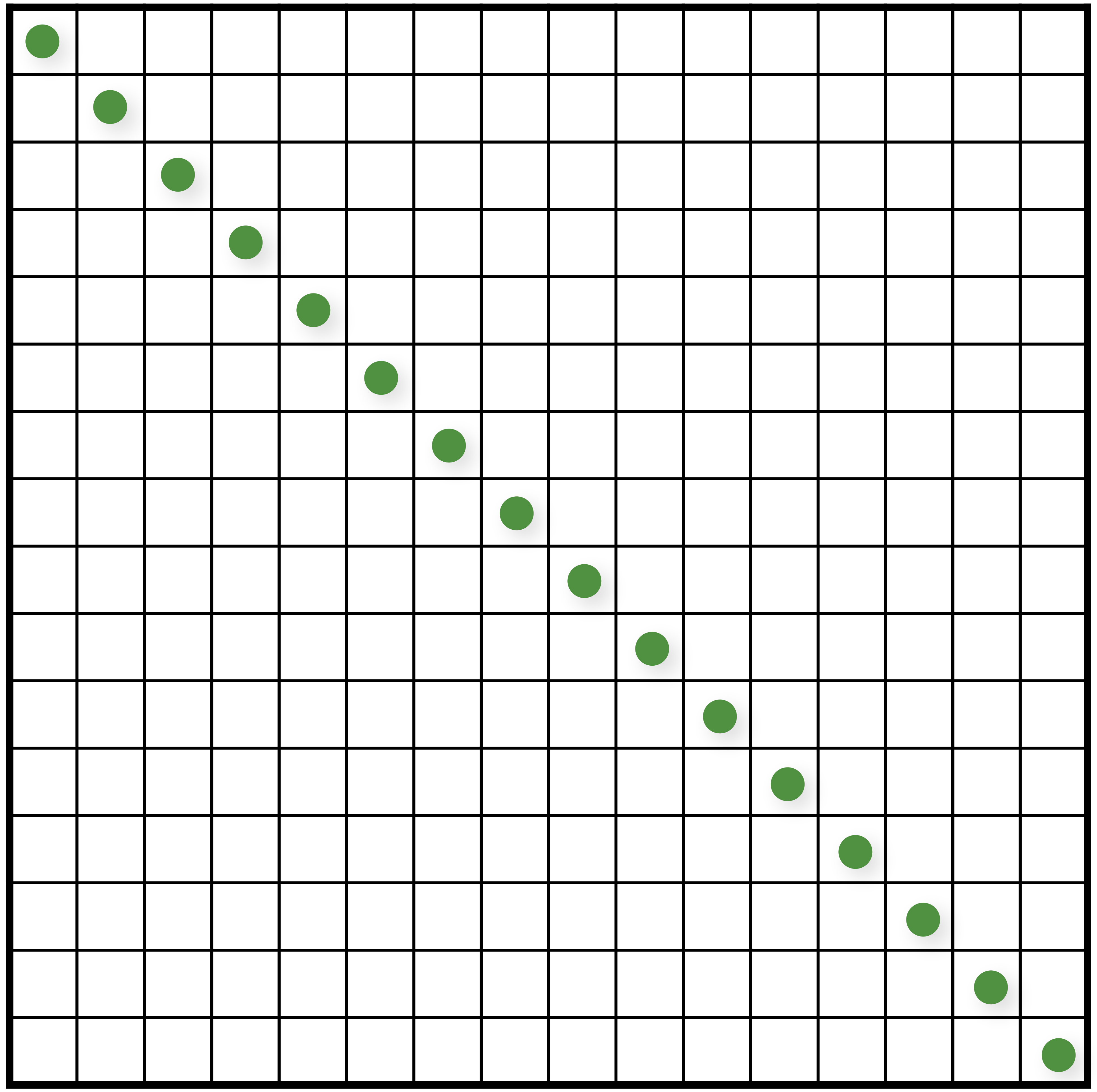

Shuffle rows

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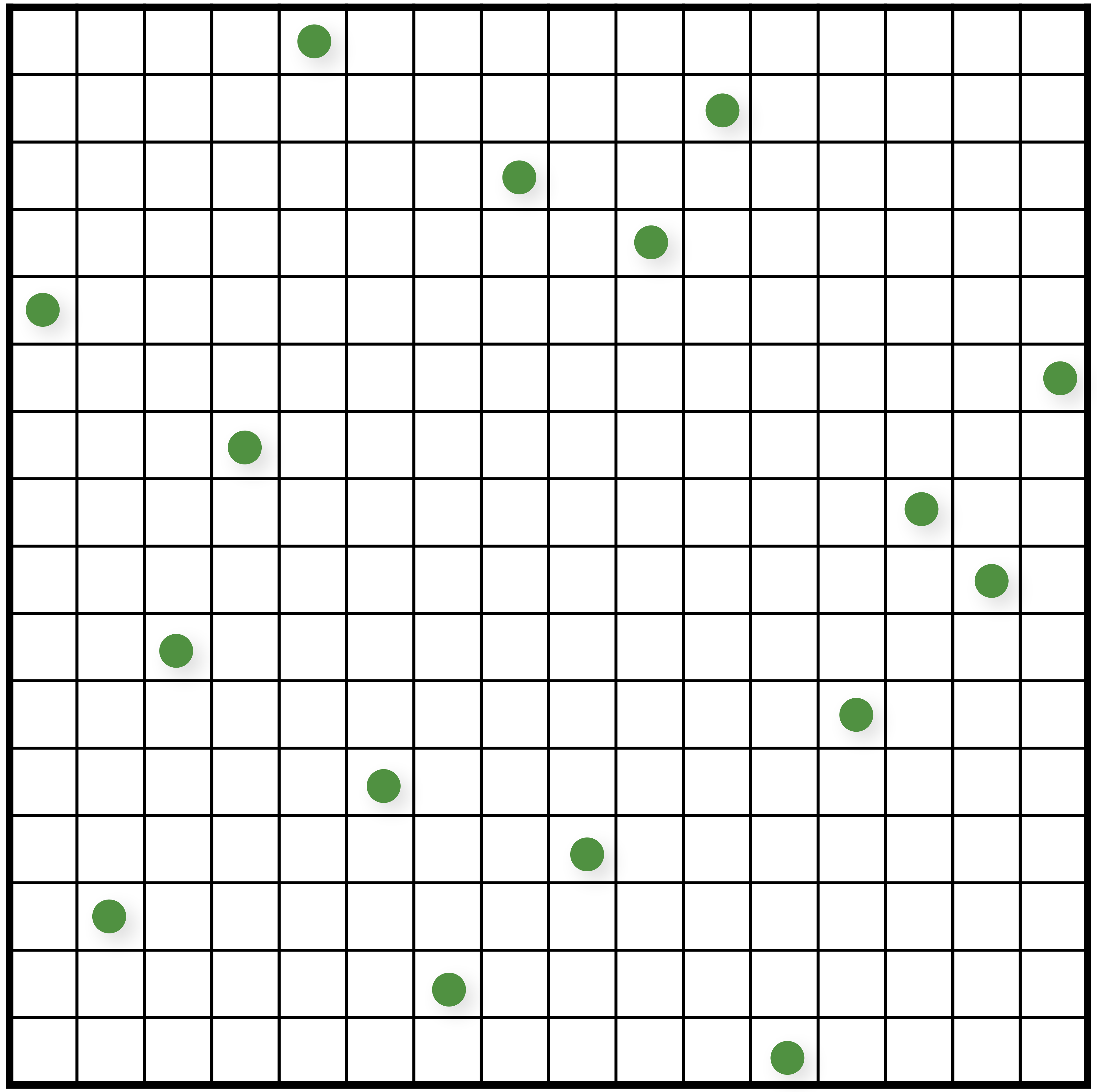

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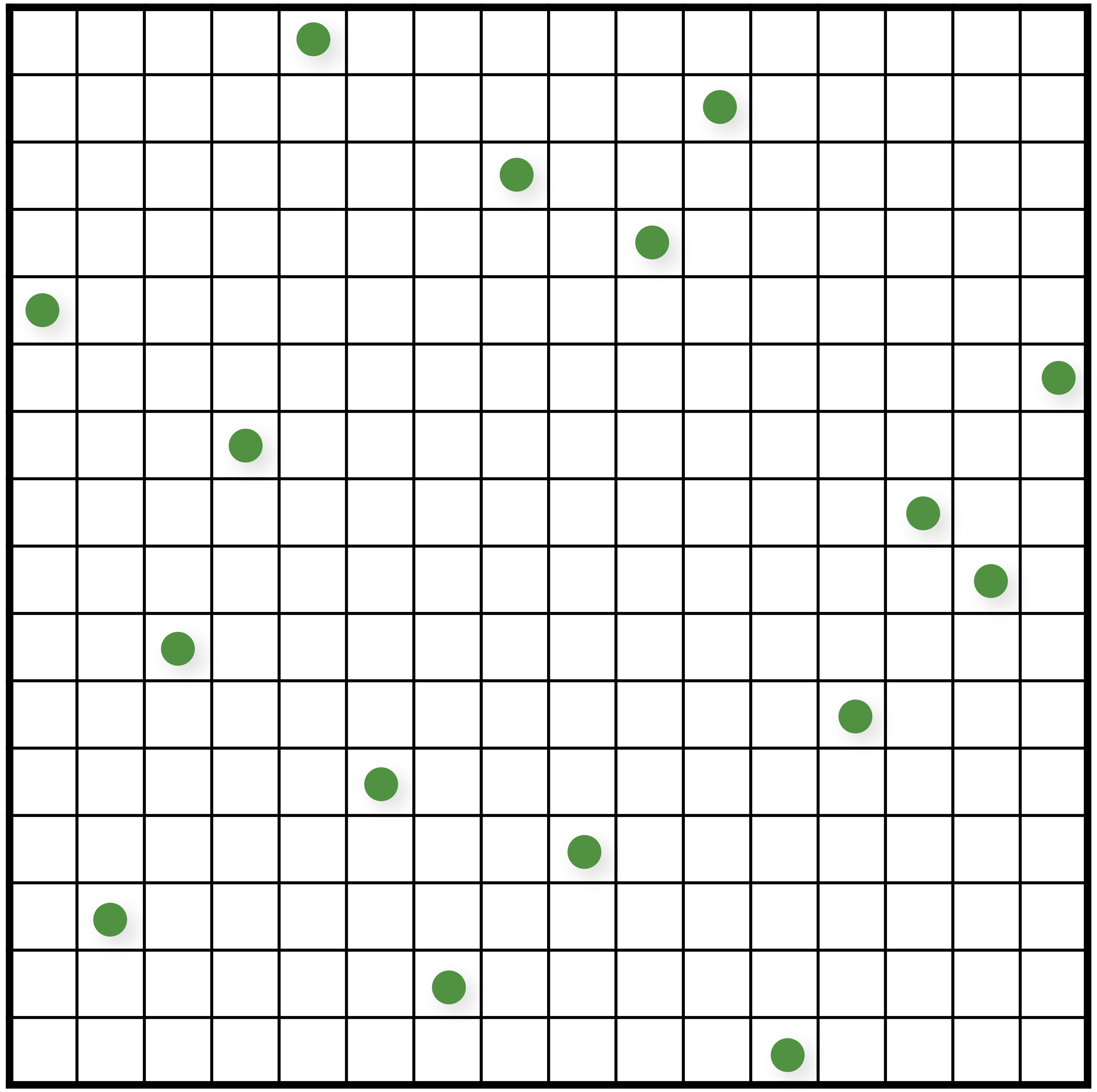

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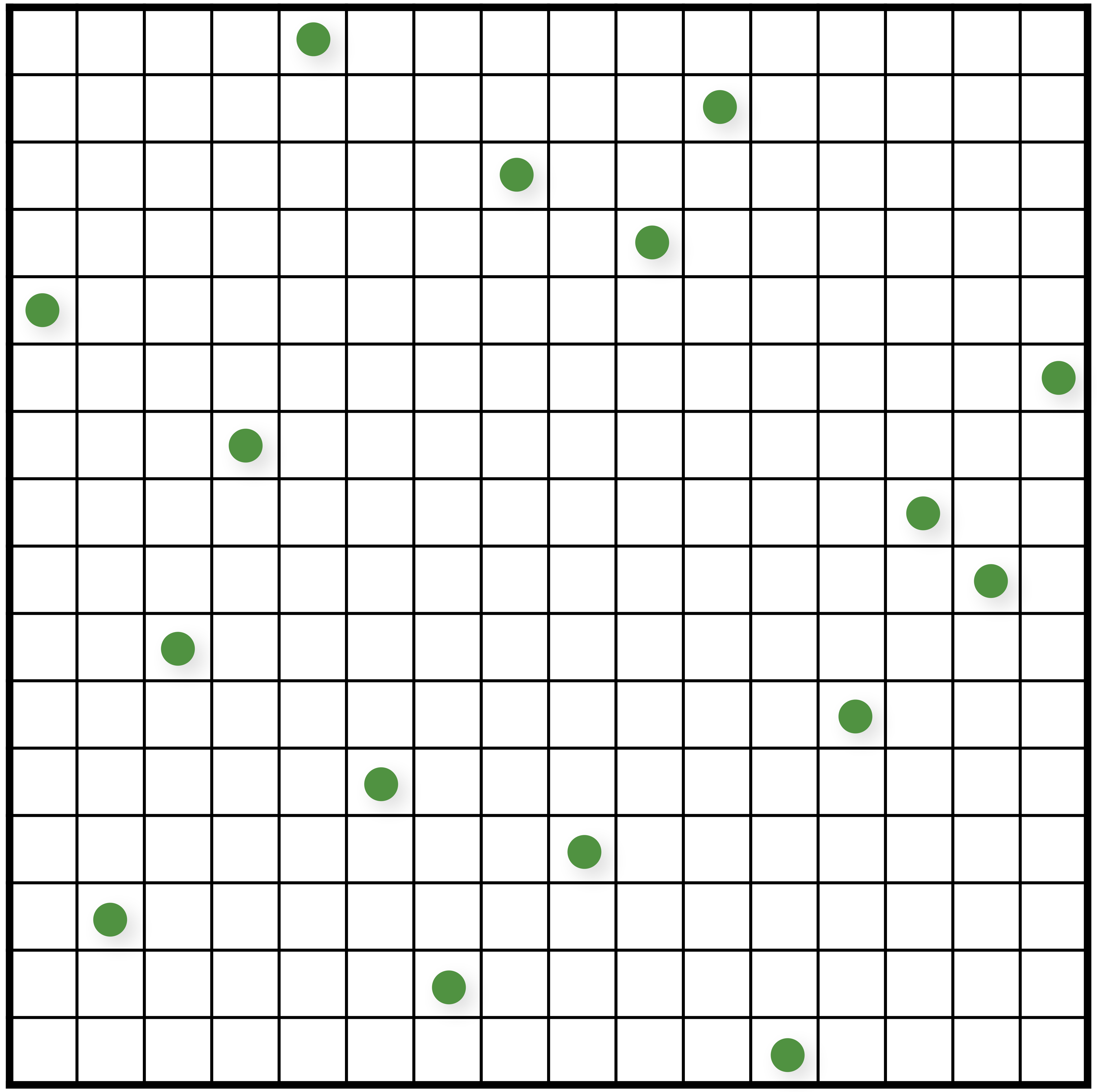

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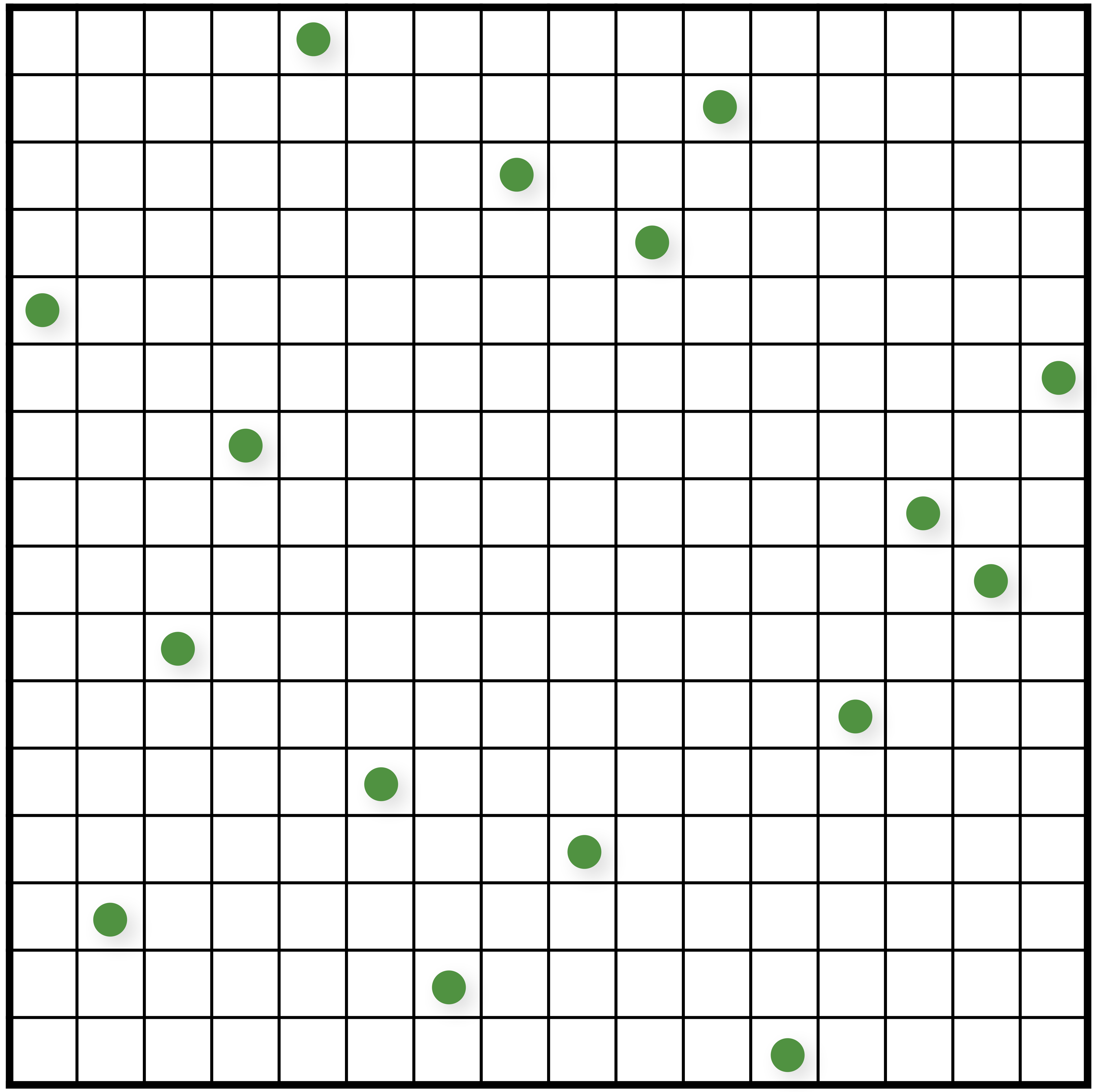
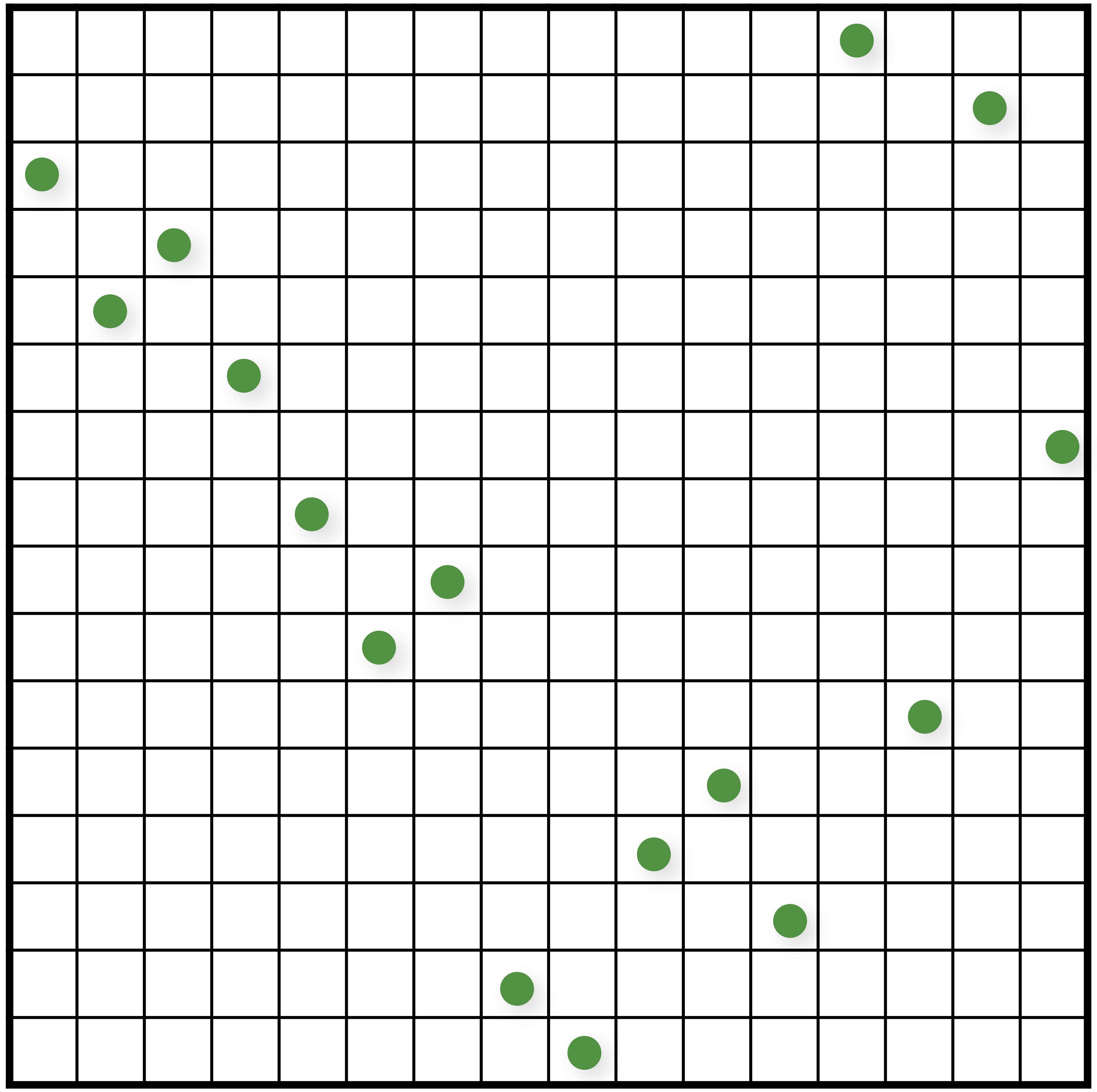


Shuffle columns

Latin Hypercube (N-Rooks) Sampling

```
// initialize the diagonal
for (uint d = 0; d < numDimensions; d++)
    for (uint i = 0; i < numS; i++)
        samples(d,i) = (i + randf())/numS;

// shuffle each dimension independently
for (uint d = 0; d < numDimensions; d++)
    shuffle(samples(d,:));
```



Shuffle columns

Latin Hypercube (N-Rooks) Sampling

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for (uint d = 0; d < numDimensions; d++)
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for (uint d = 0; d < numDimensions; d++)
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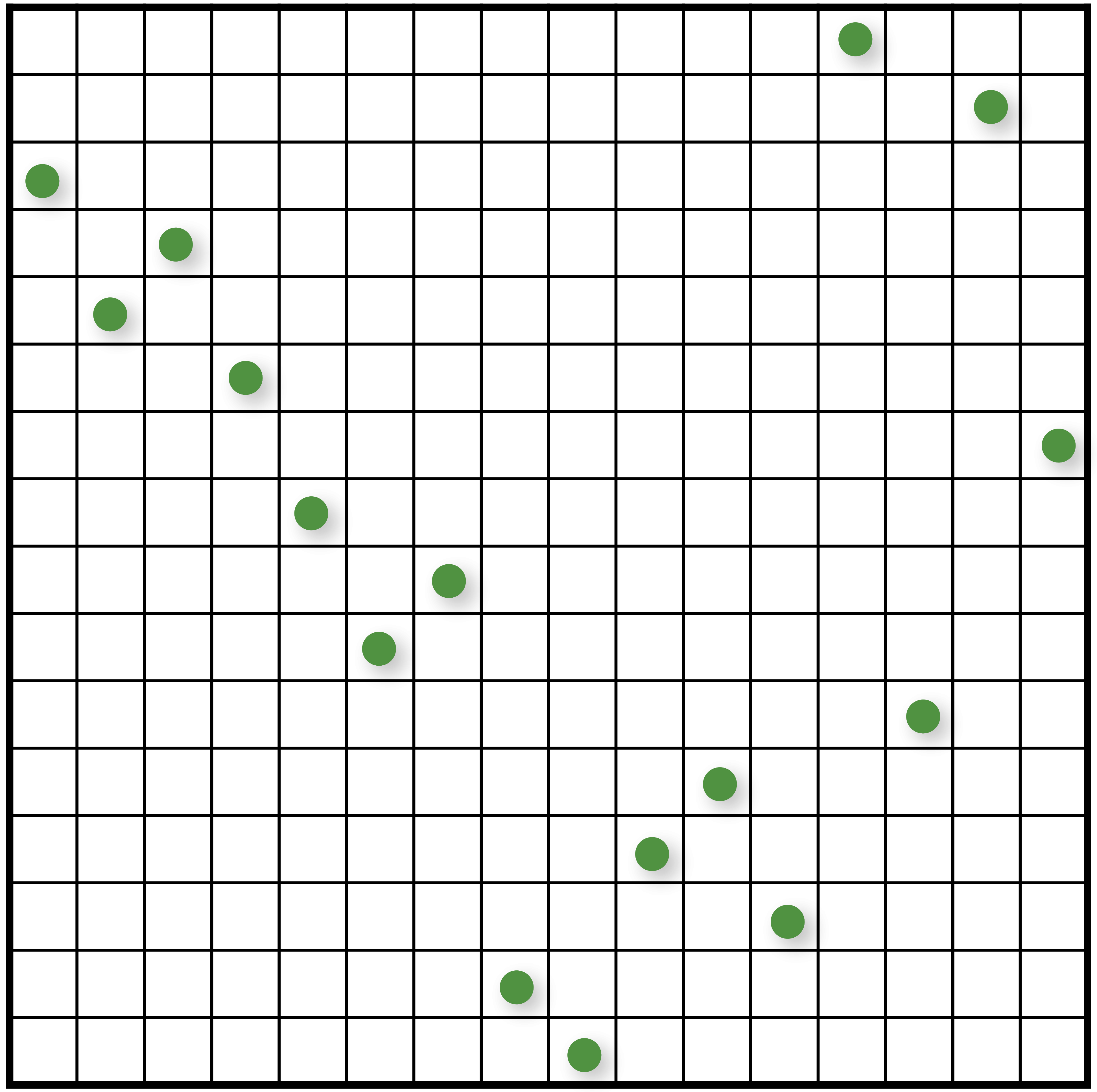
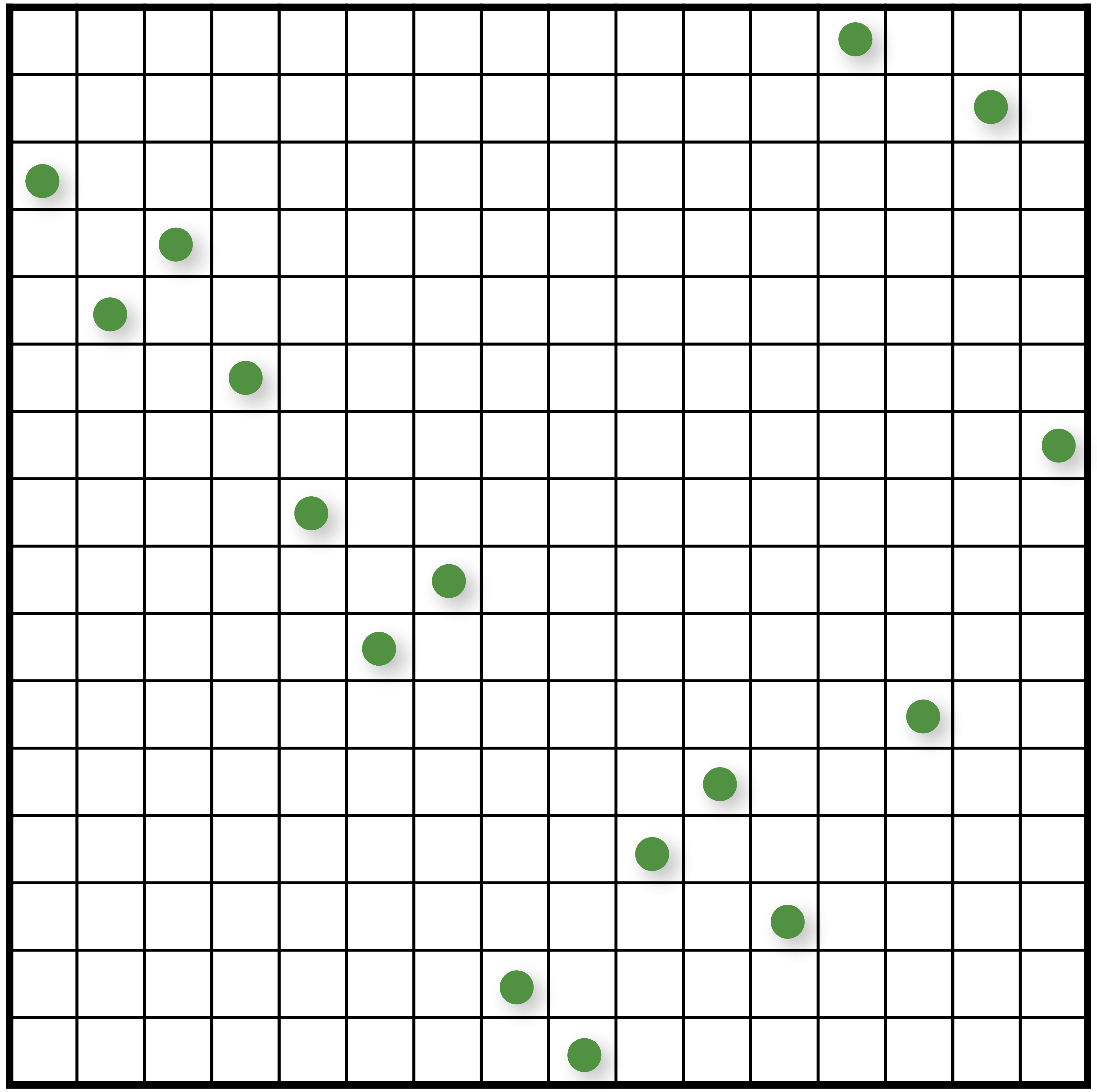
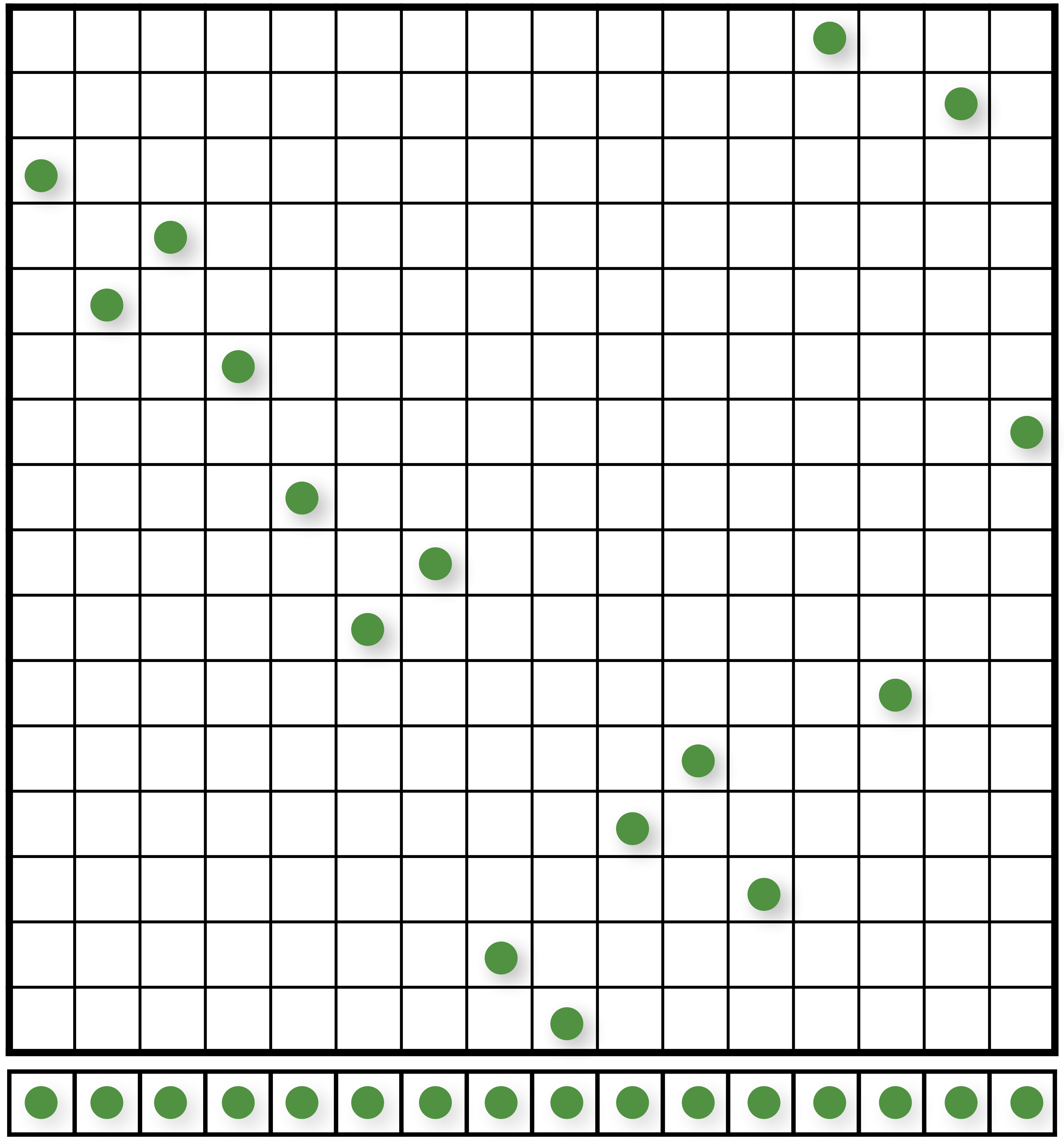
```



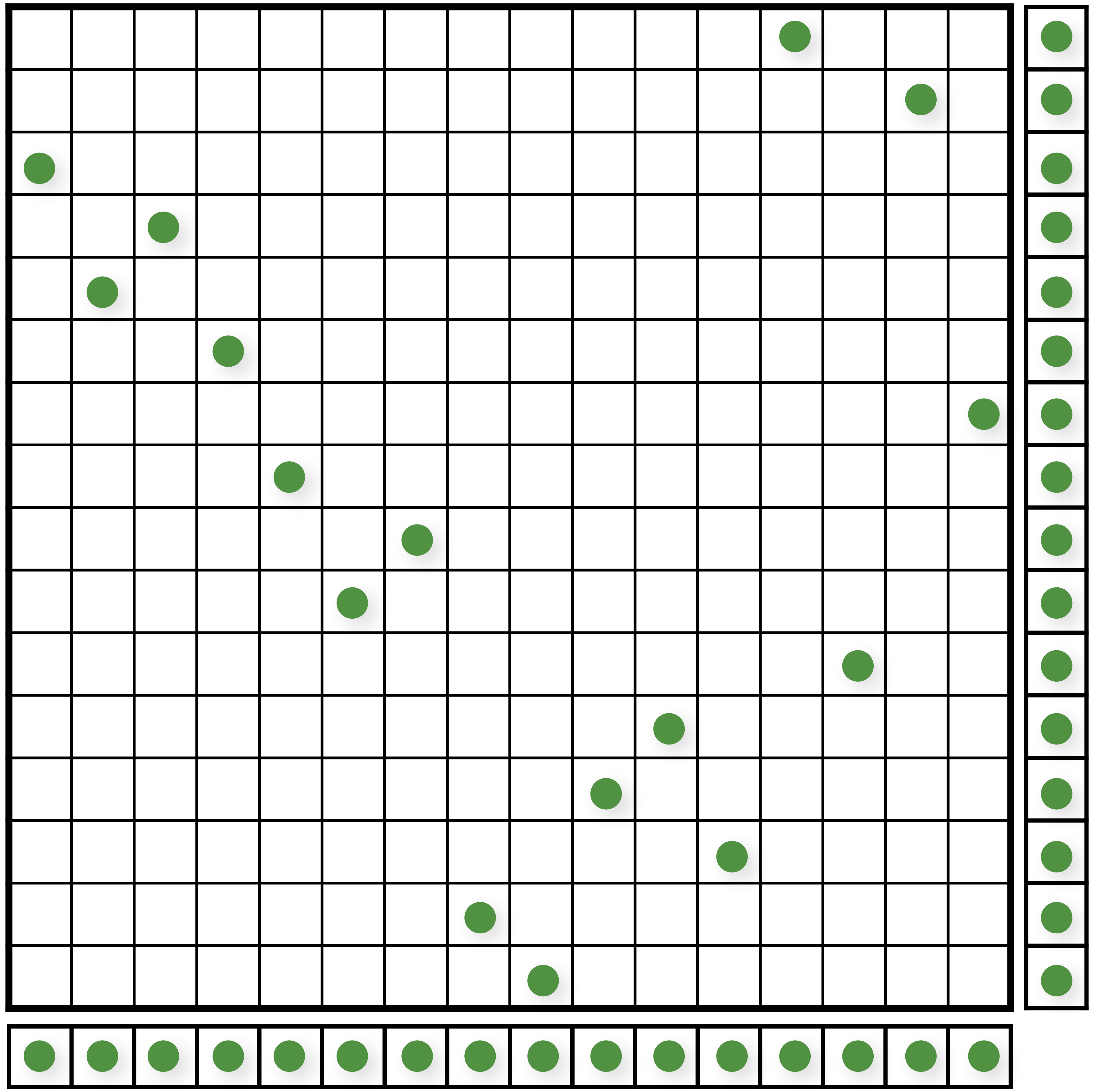
Latin Hypercube (N-Rooks) Sampling



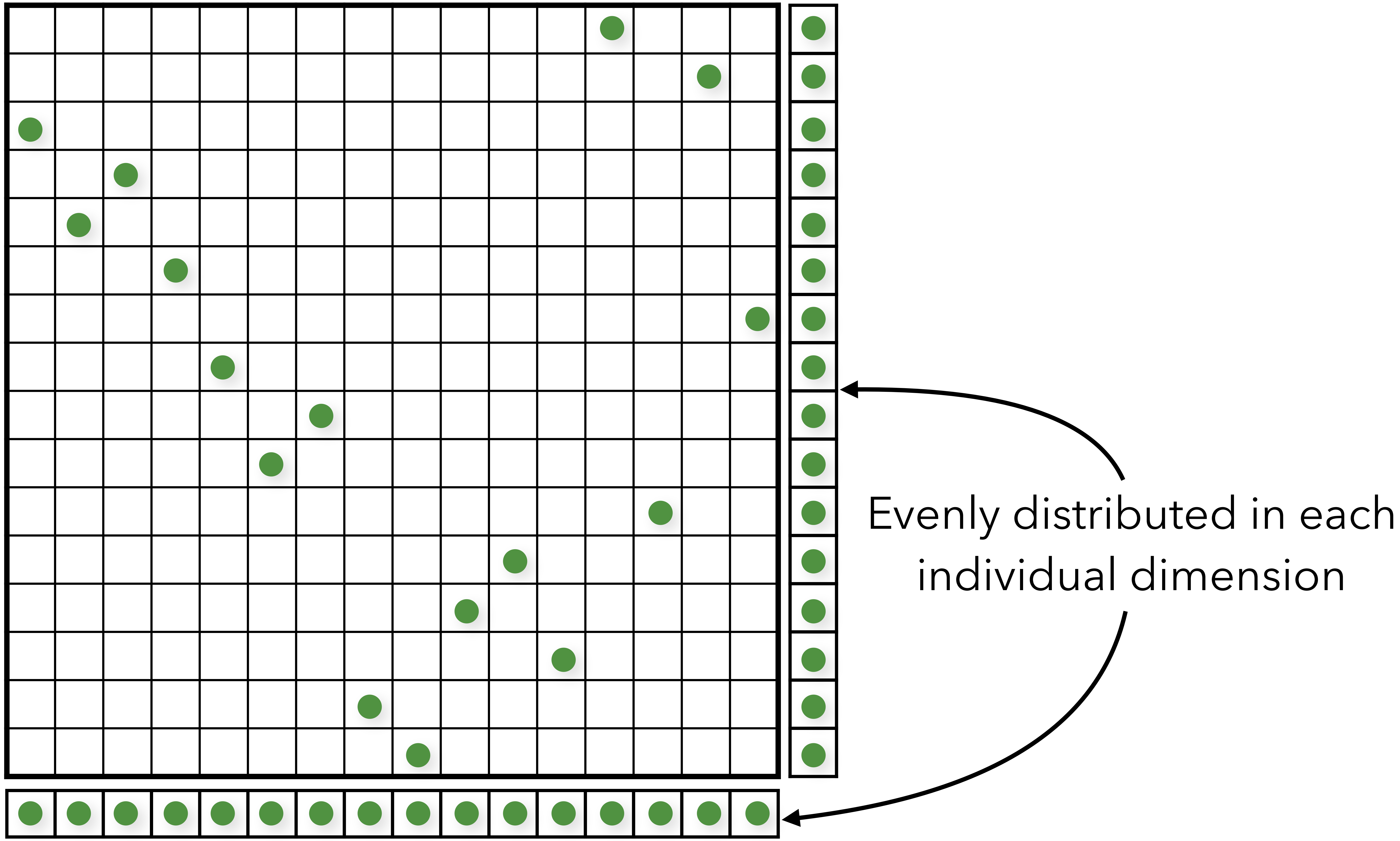
Latin Hypercube (N-Rooks) Sampling



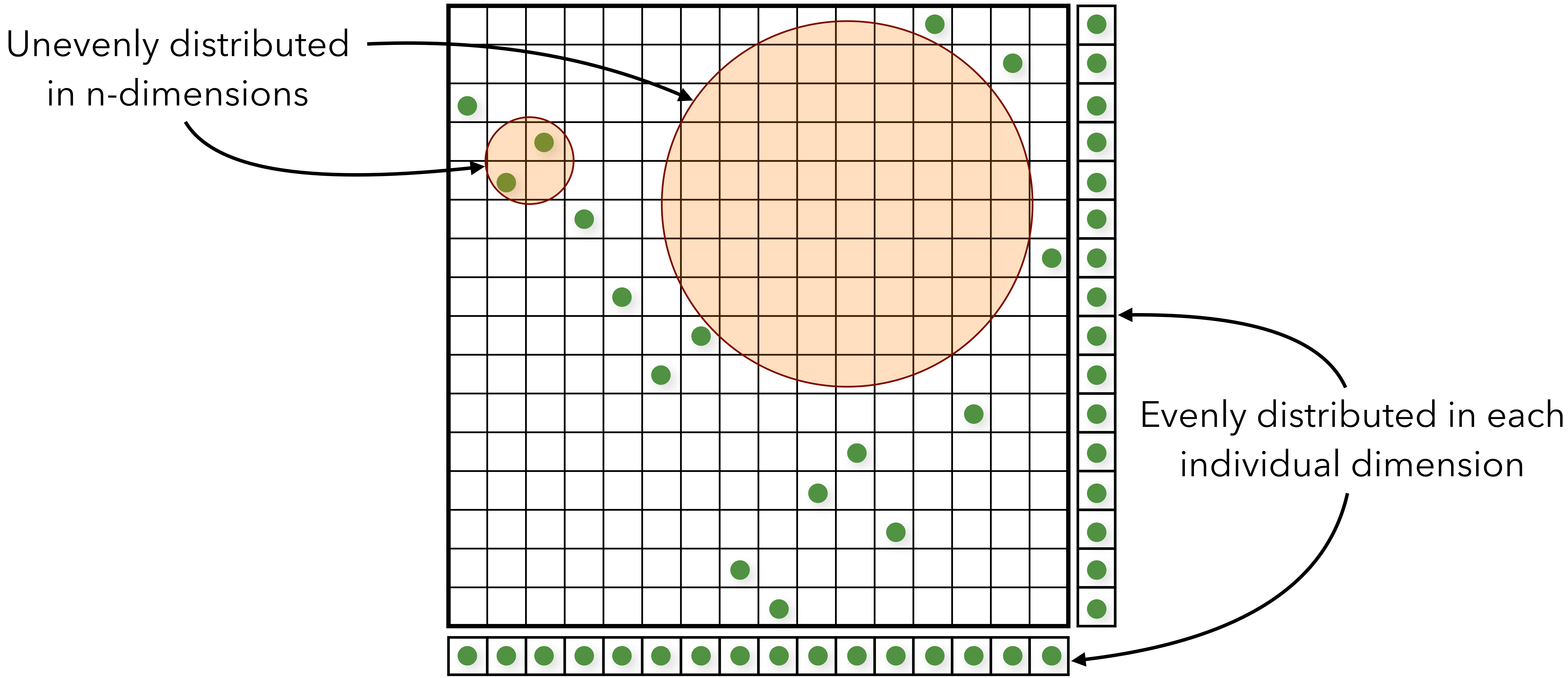
Latin Hypercube (N-Rooks) Sampling



Latin Hypercube (N-Rooks) Sampling

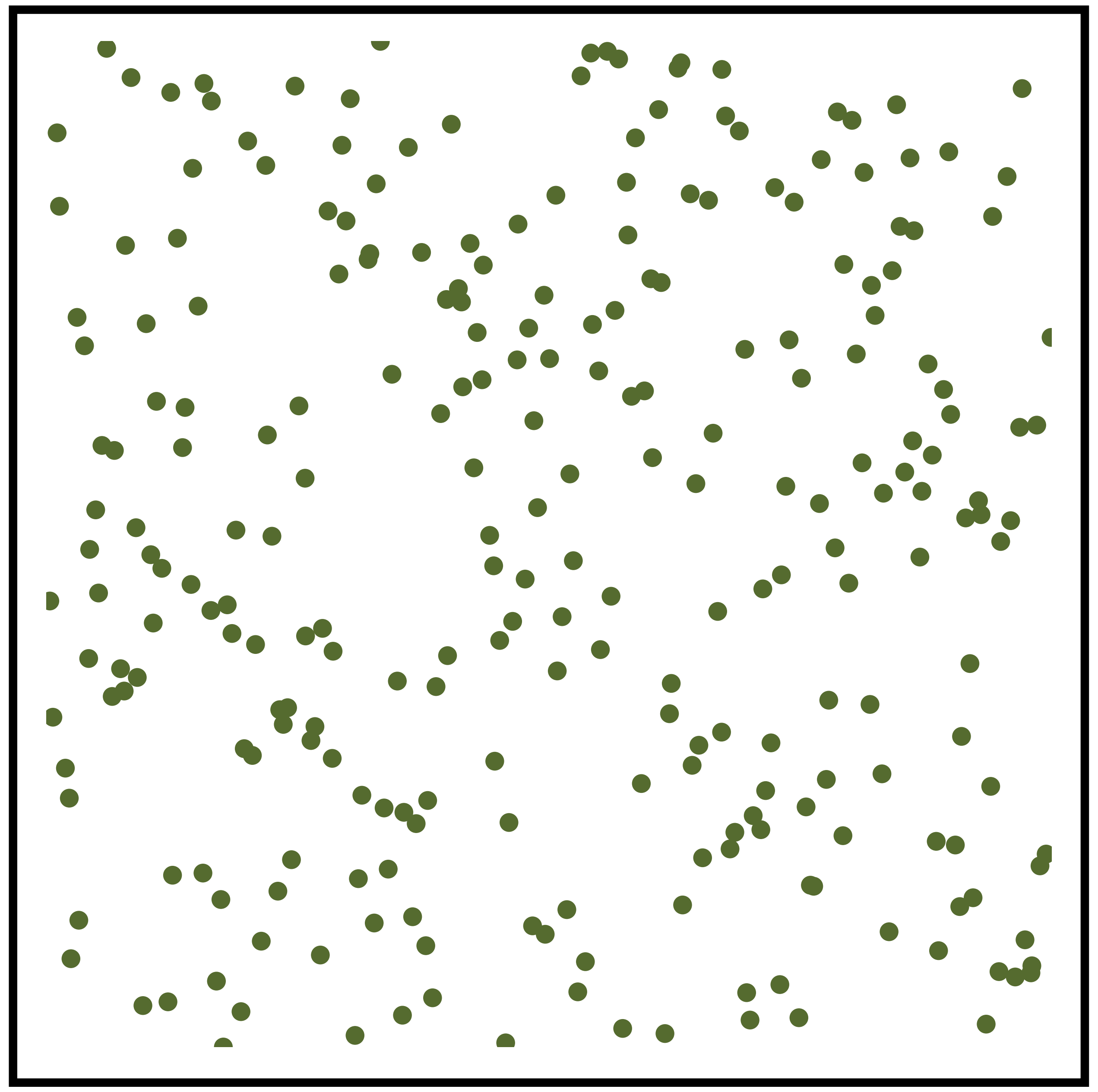


Latin Hypercube (N-Rooks) Sampling



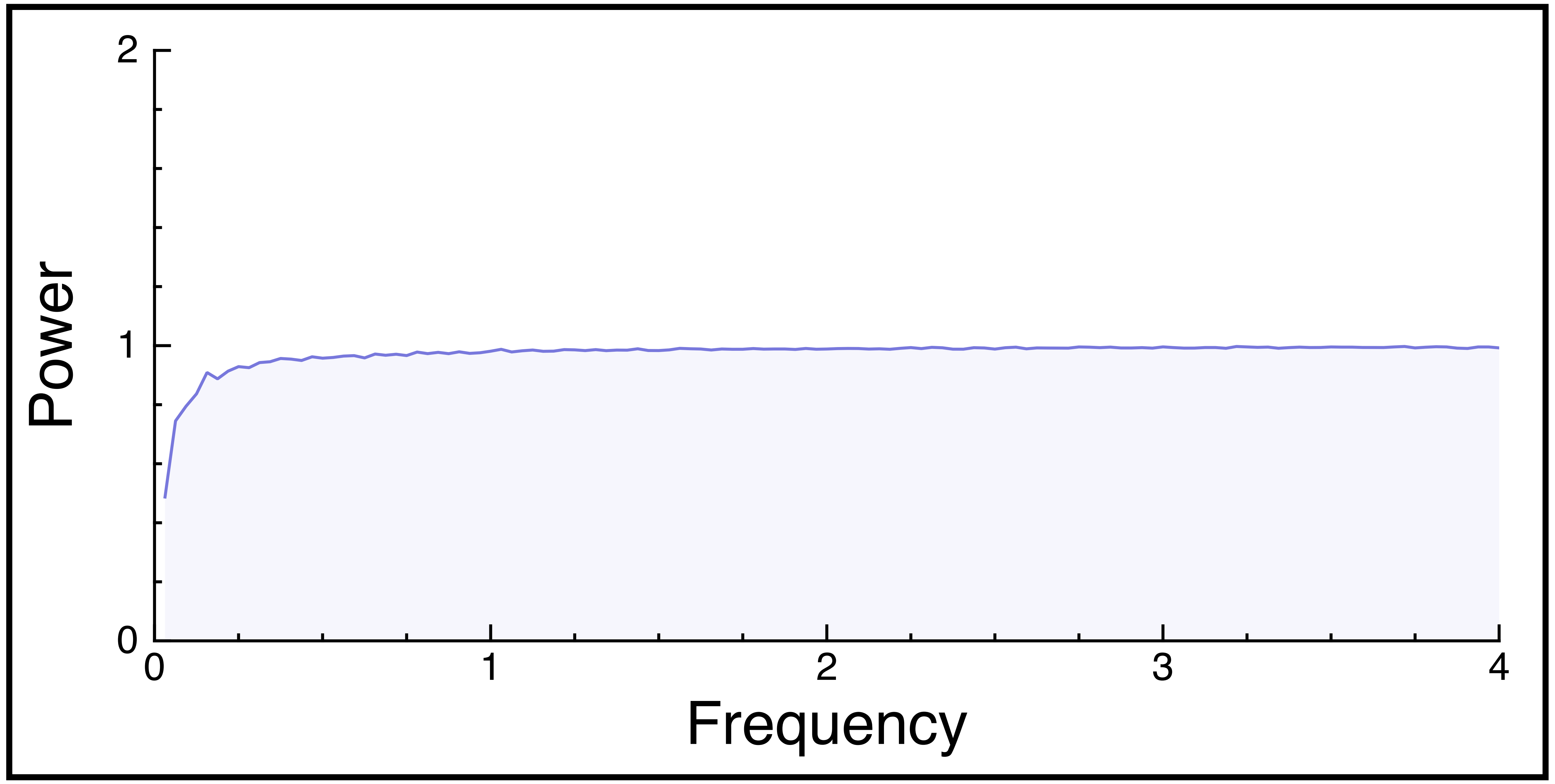
N-Rooks Sampling

Samples



Expected power spectrum

Radial mean

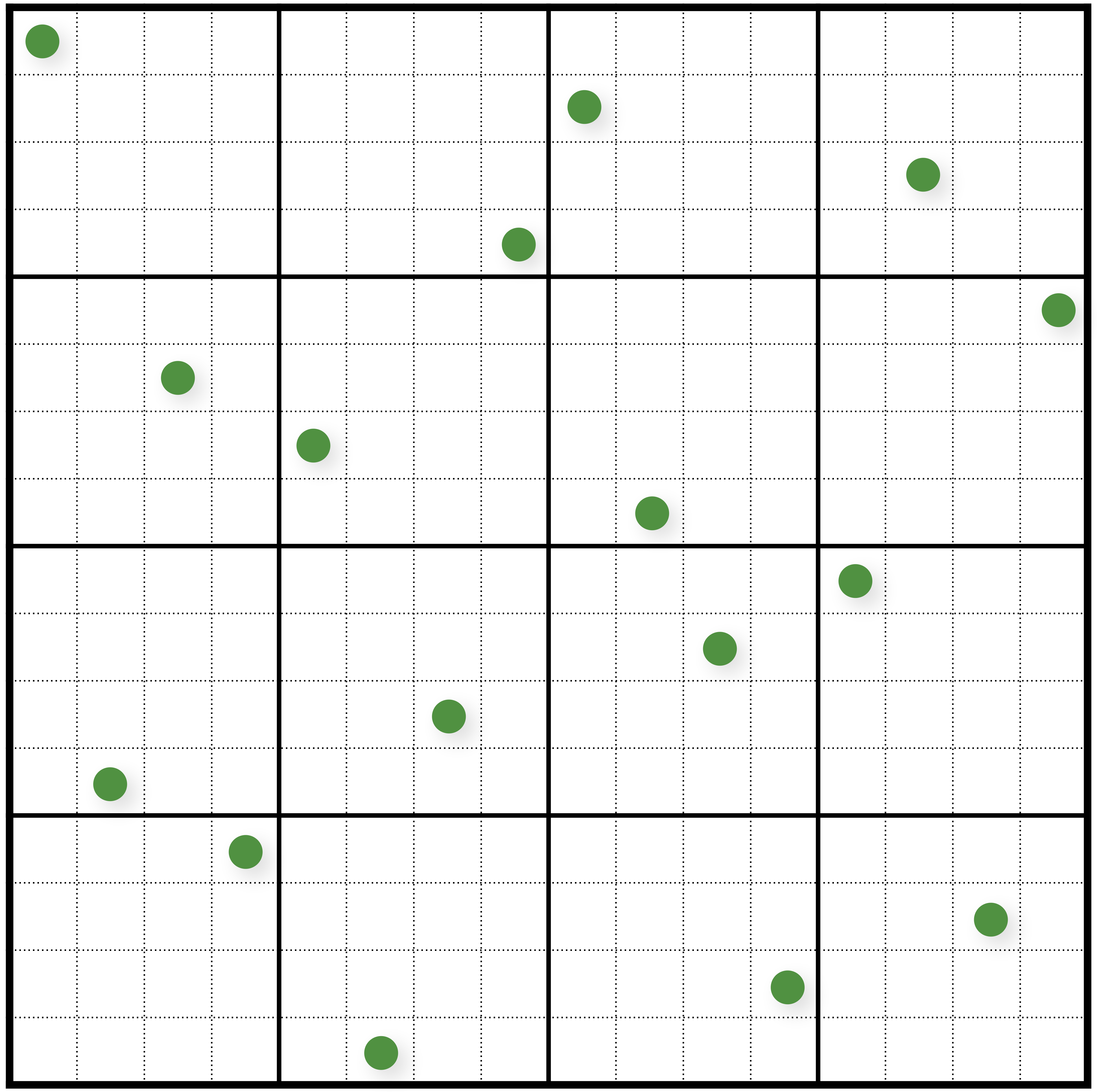


Multi-Jittered Sampling

Kenneth Chiu, Peter Shirley, and Changyaw Wang. "Multi-jittered sampling." In *Graphics Gems IV*, pp. 370–374. Academic Press, May 1994.

- combine N-Rooks and Jittered stratification constraints

Multi-Jittered Sampling



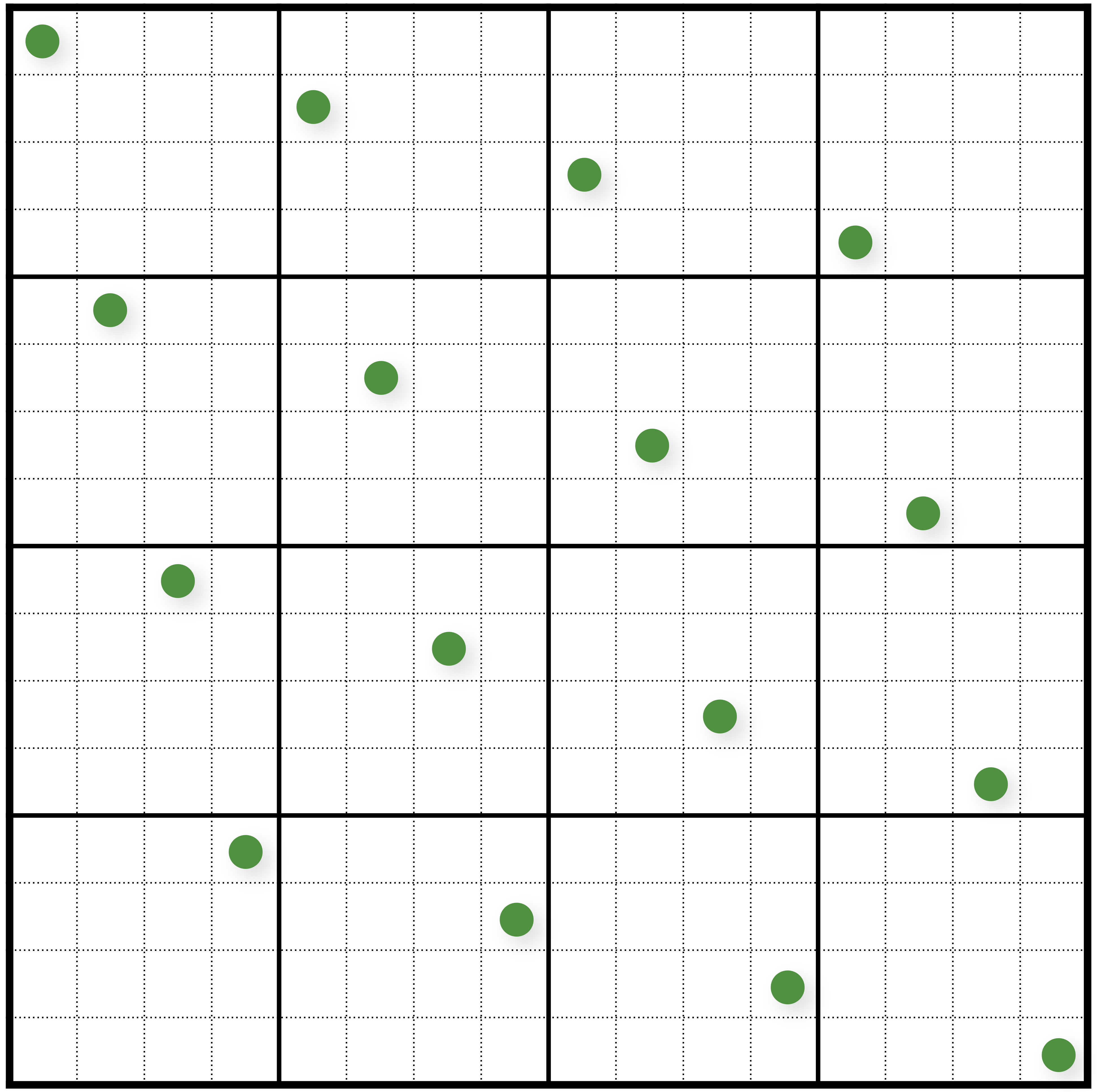
Multi-Jittered Sampling

```
// initialize
float cellSize = 1.0 / (resX*resY);
for (uint i = 0; i < resX; i++)
    for (uint j = 0; j < resY; j++)
    {
        samples(i,j).x = i/resX + (j+randf()) / (resX*resY);
        samples(i,j).y = j/resY + (i+randf()) / (resX*resY);
    }

// shuffle x coordinates within each column of cells
for (uint i = 0; i < resX; i++)
    for (uint j = resY-1; j >= 1; j--)
        swap(samples(i, j).x, samples(i, randi(0, j)).x);

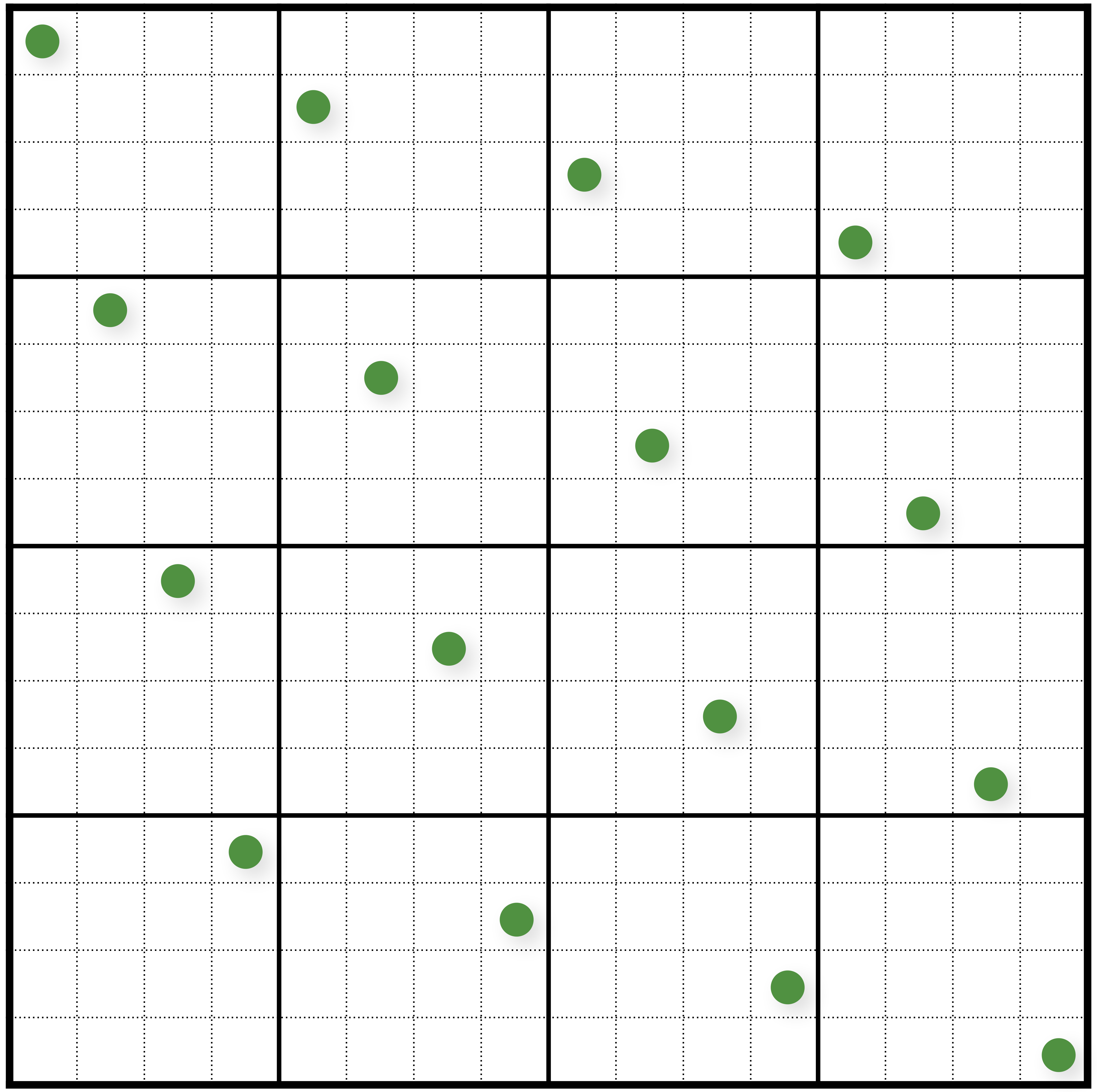
// shuffle y coordinates within each row of cells
for (unsigned j = 0; j < resY; j++)
    for (unsigned i = resX-1; i >= 1; i--)
        swap(samples(i, j).y, samples(randi(0, i), j).y);
```

Multi-Jittered Sampling



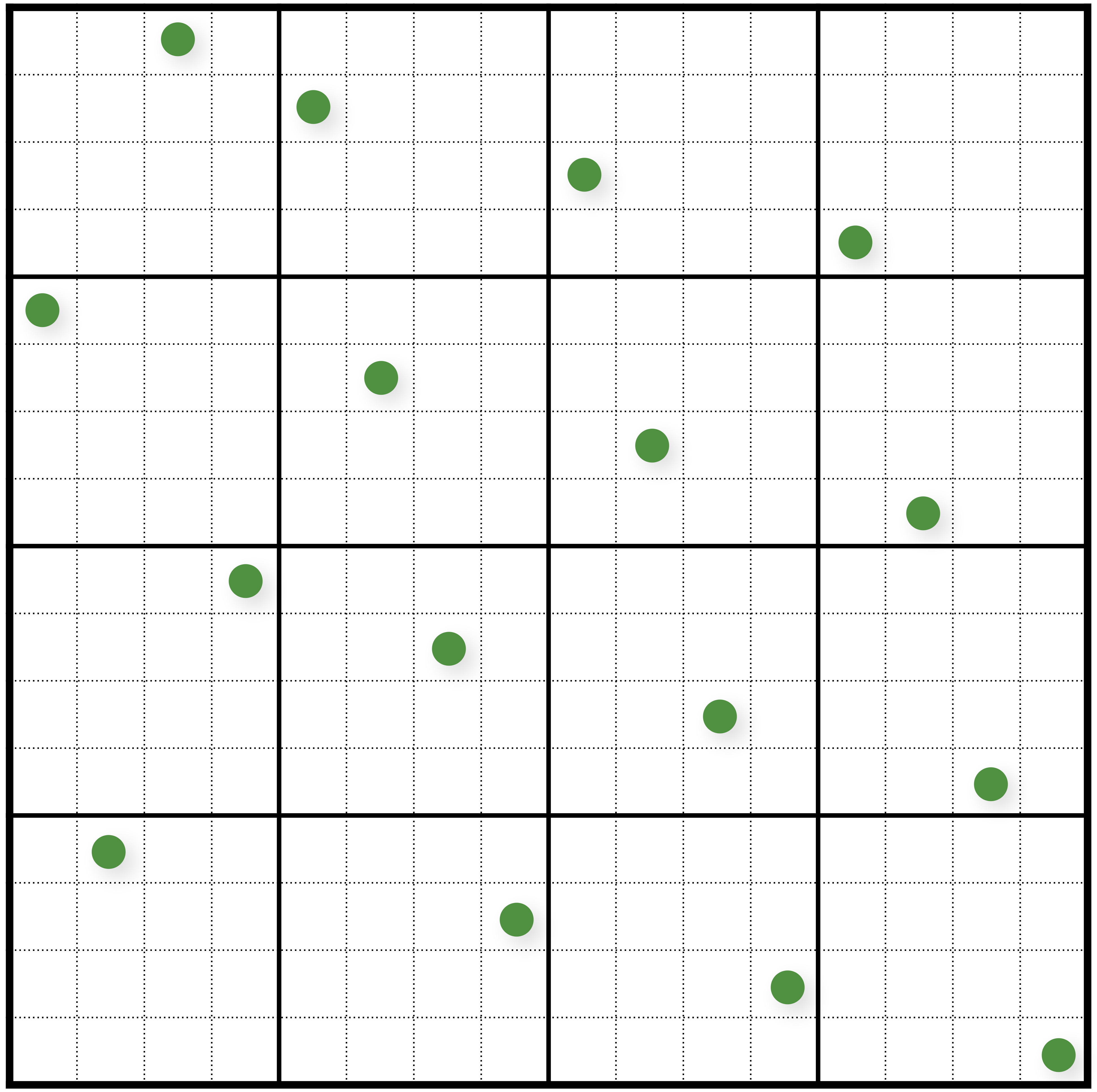
Initialize

Multi-Jittered Sampling



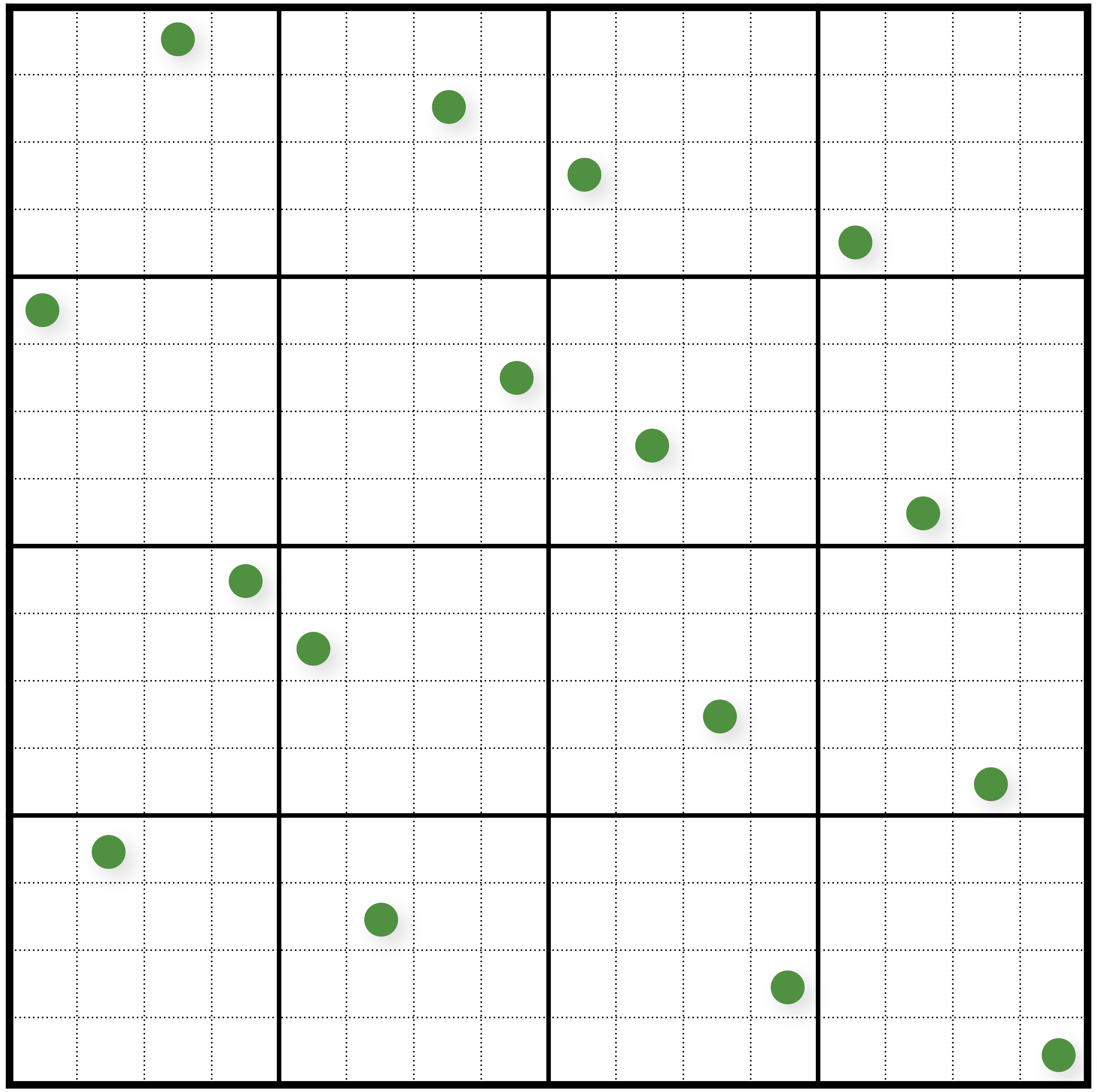
Shuffle x-coords

Multi-Jittered Sampling



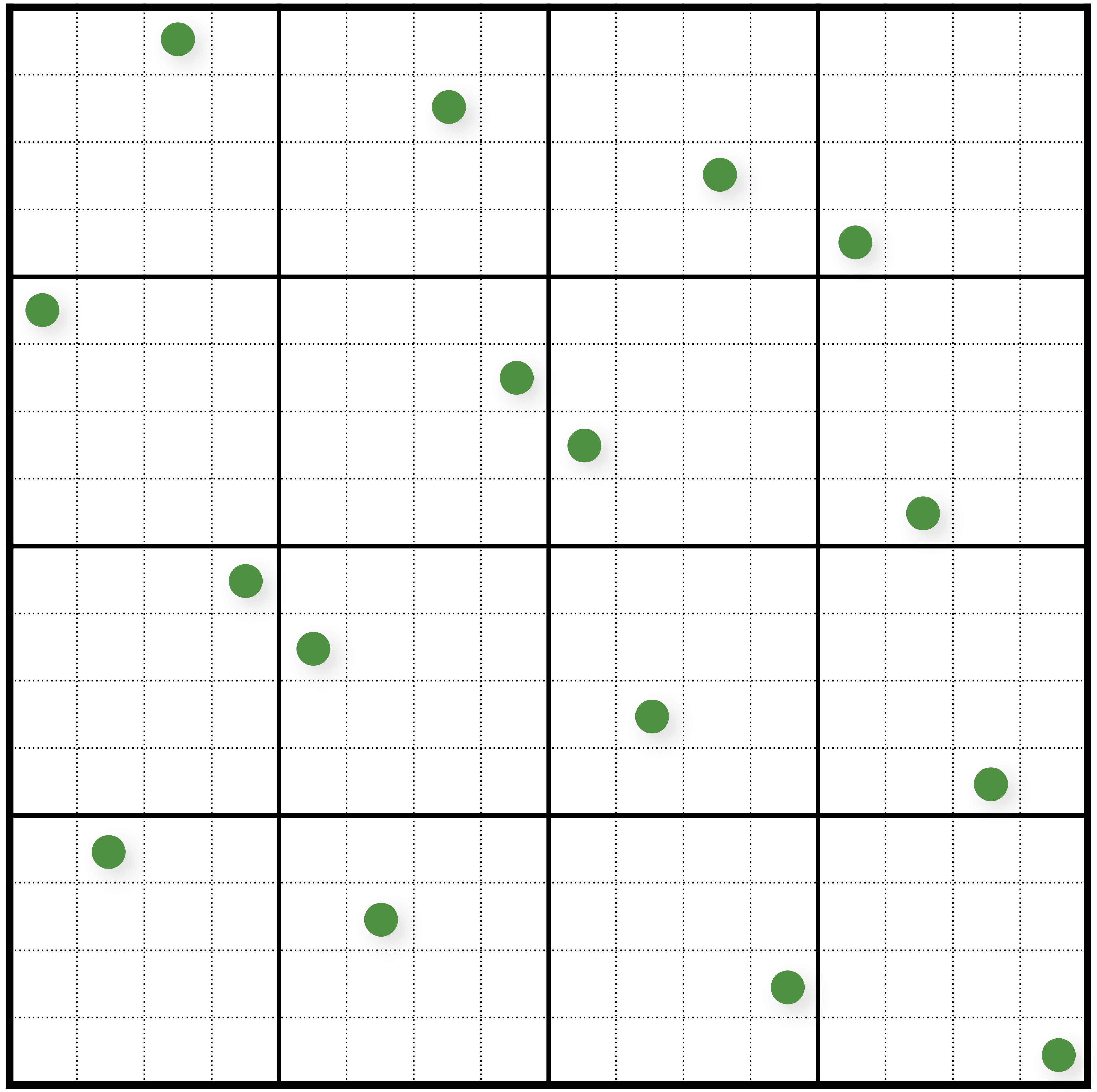
Shuffle x-coords

Multi-Jittered Sampling



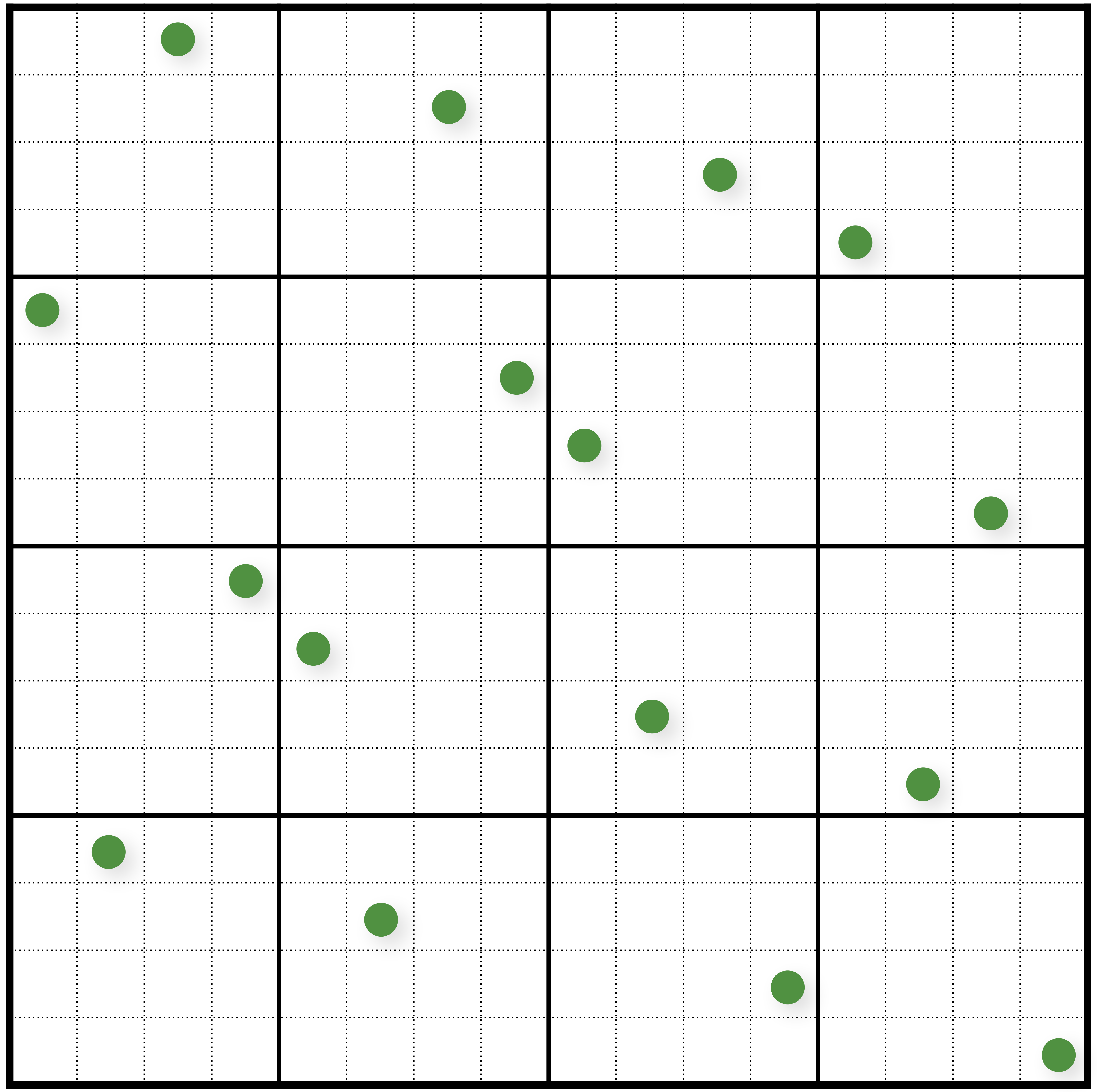
Shuffle x-coords

Multi-Jittered Sampling



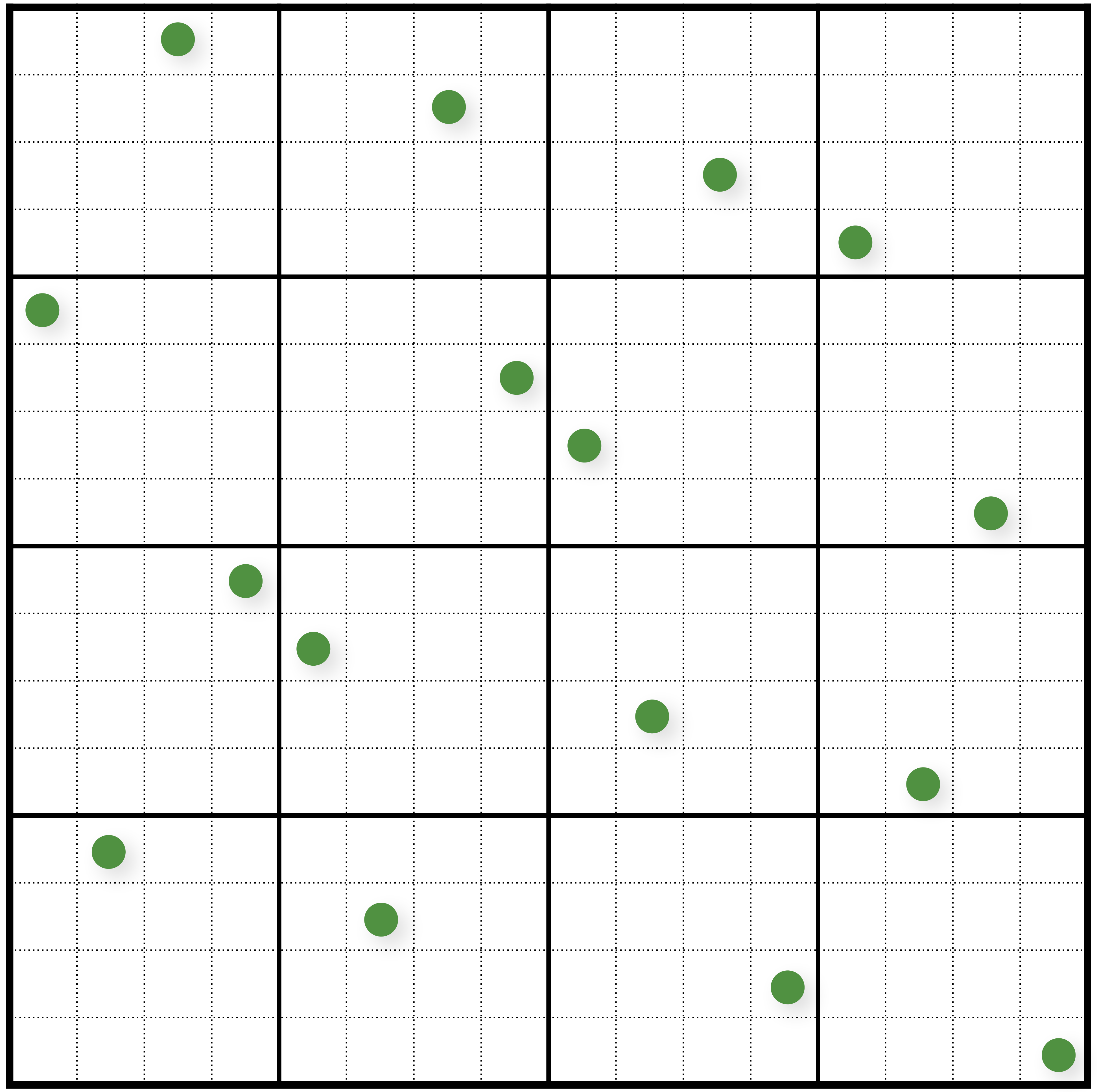
Shuffle x-coords

Multi-Jittered Sampling

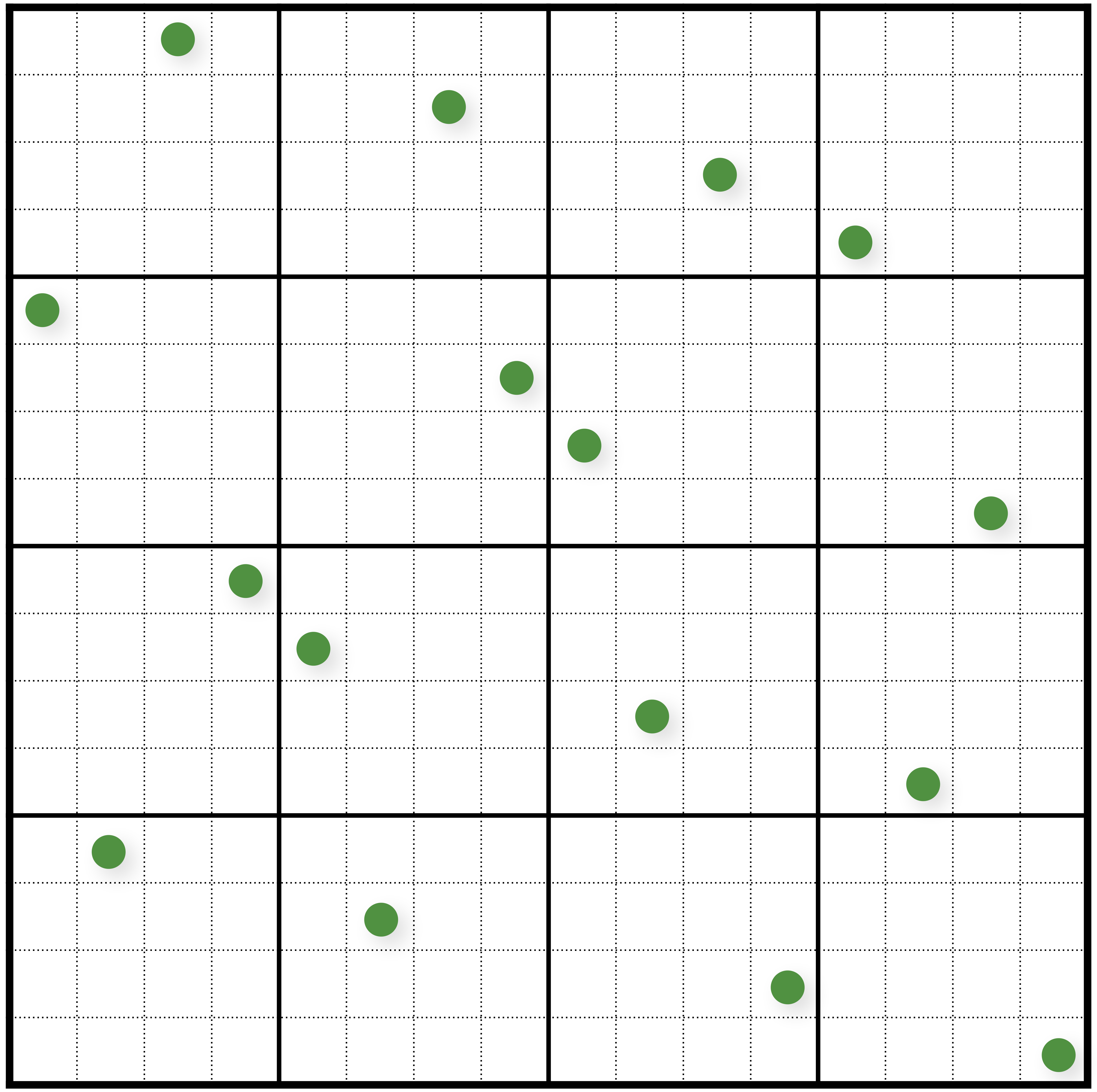


Shuffle x-coords

Multi-Jittered Sampling

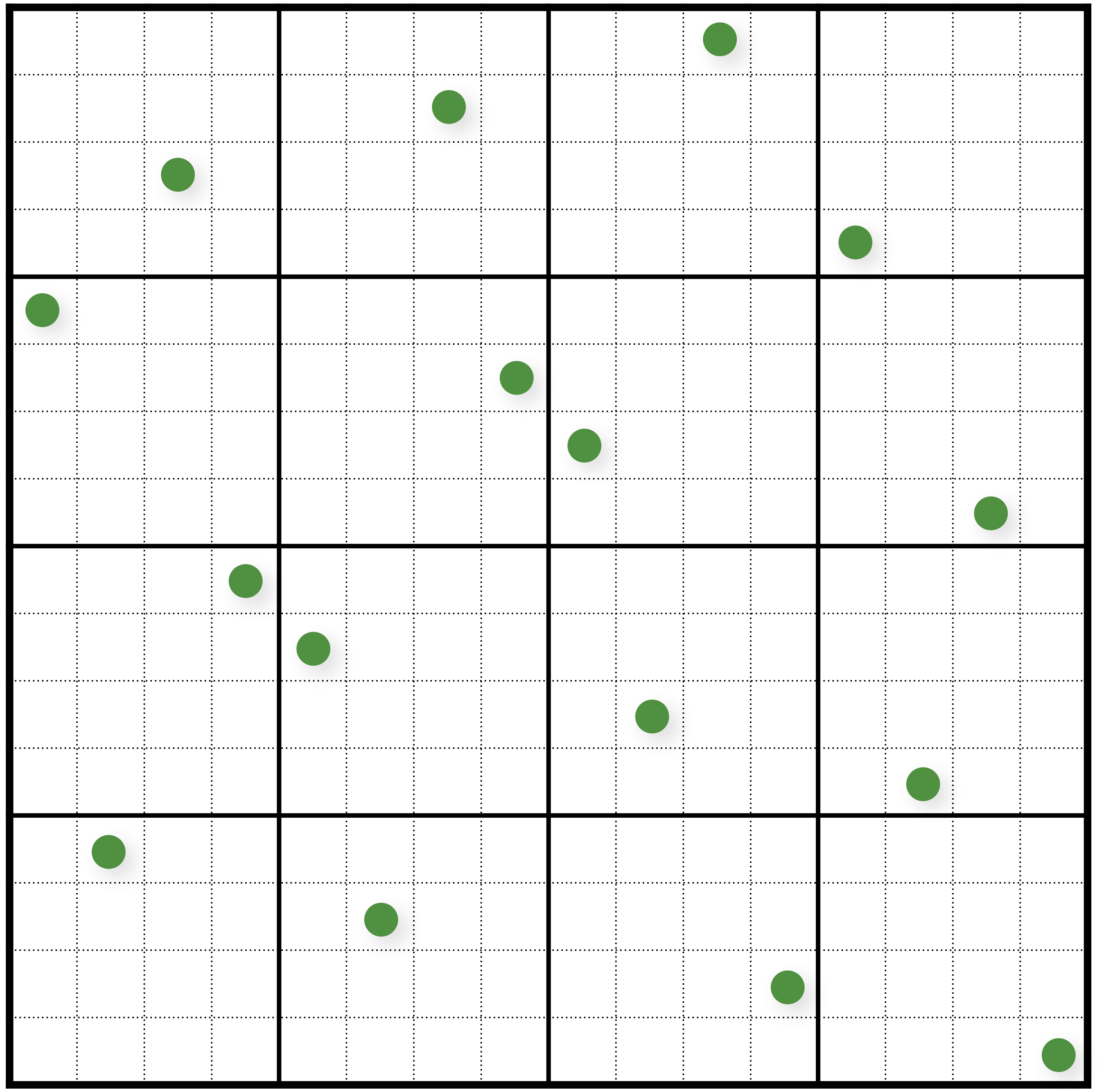


Multi-Jittered Sampling



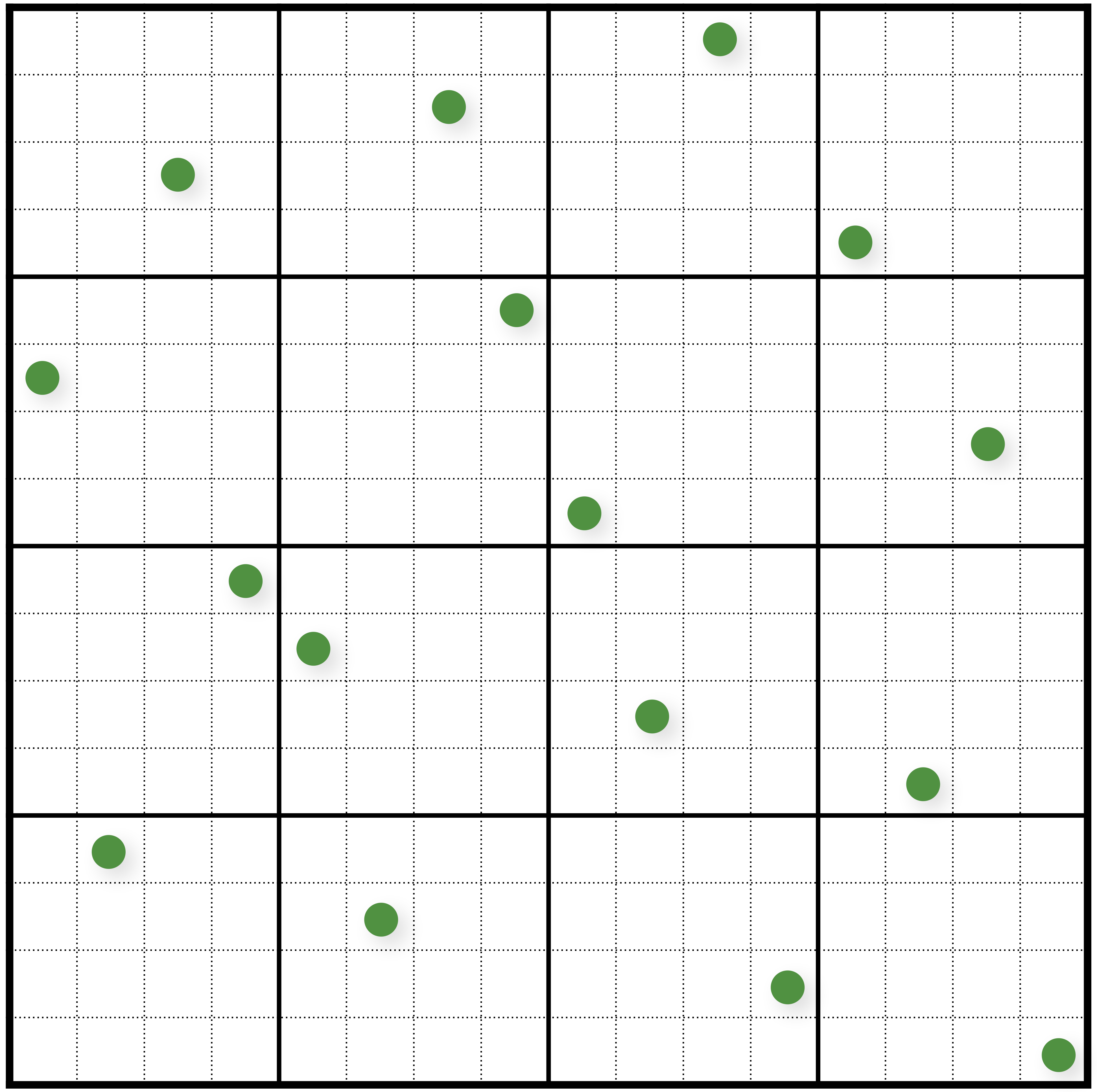
Shuffle y-coords

Multi-Jittered Sampling



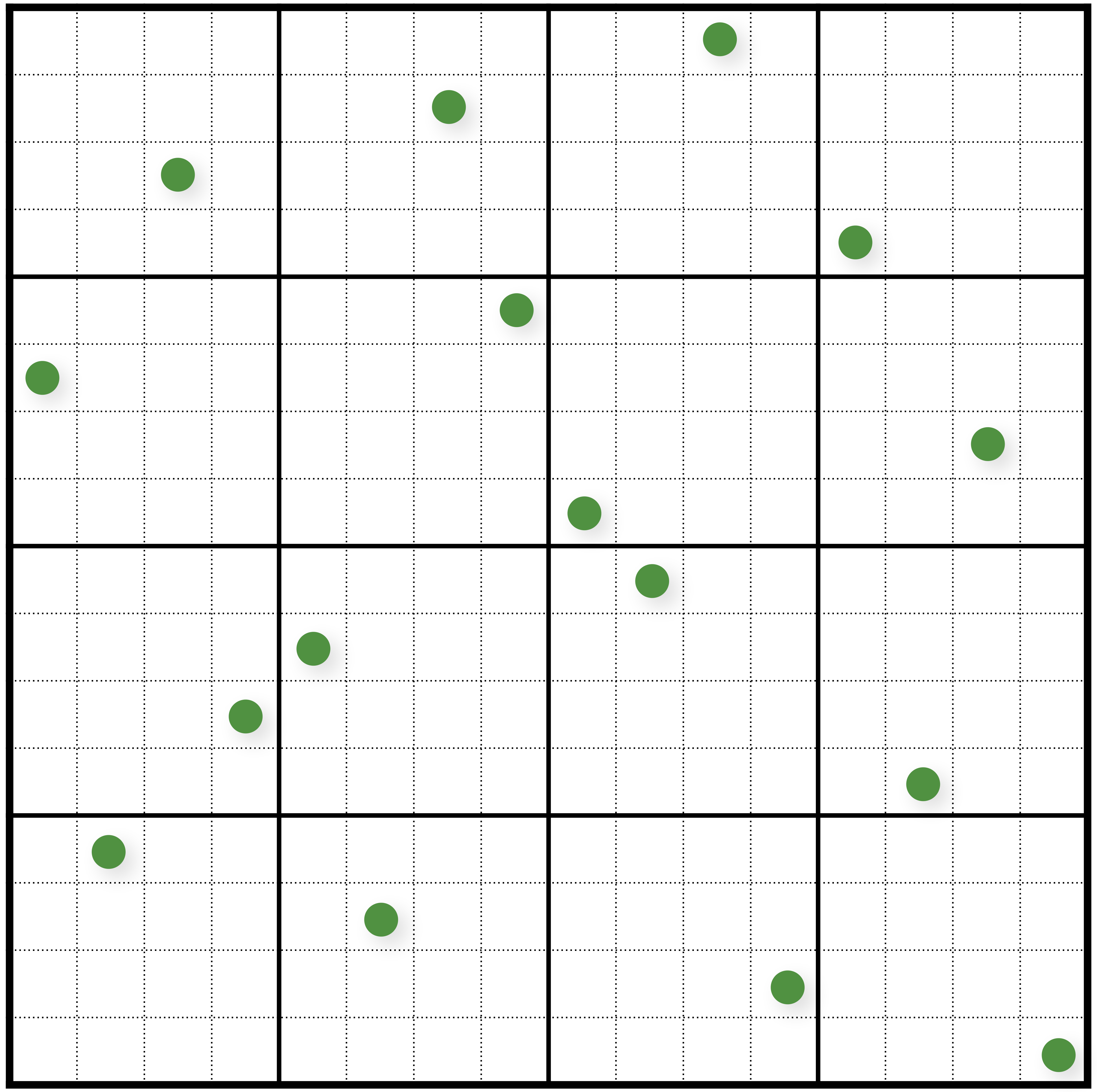
Shuffle y-coords

Multi-Jittered Sampling



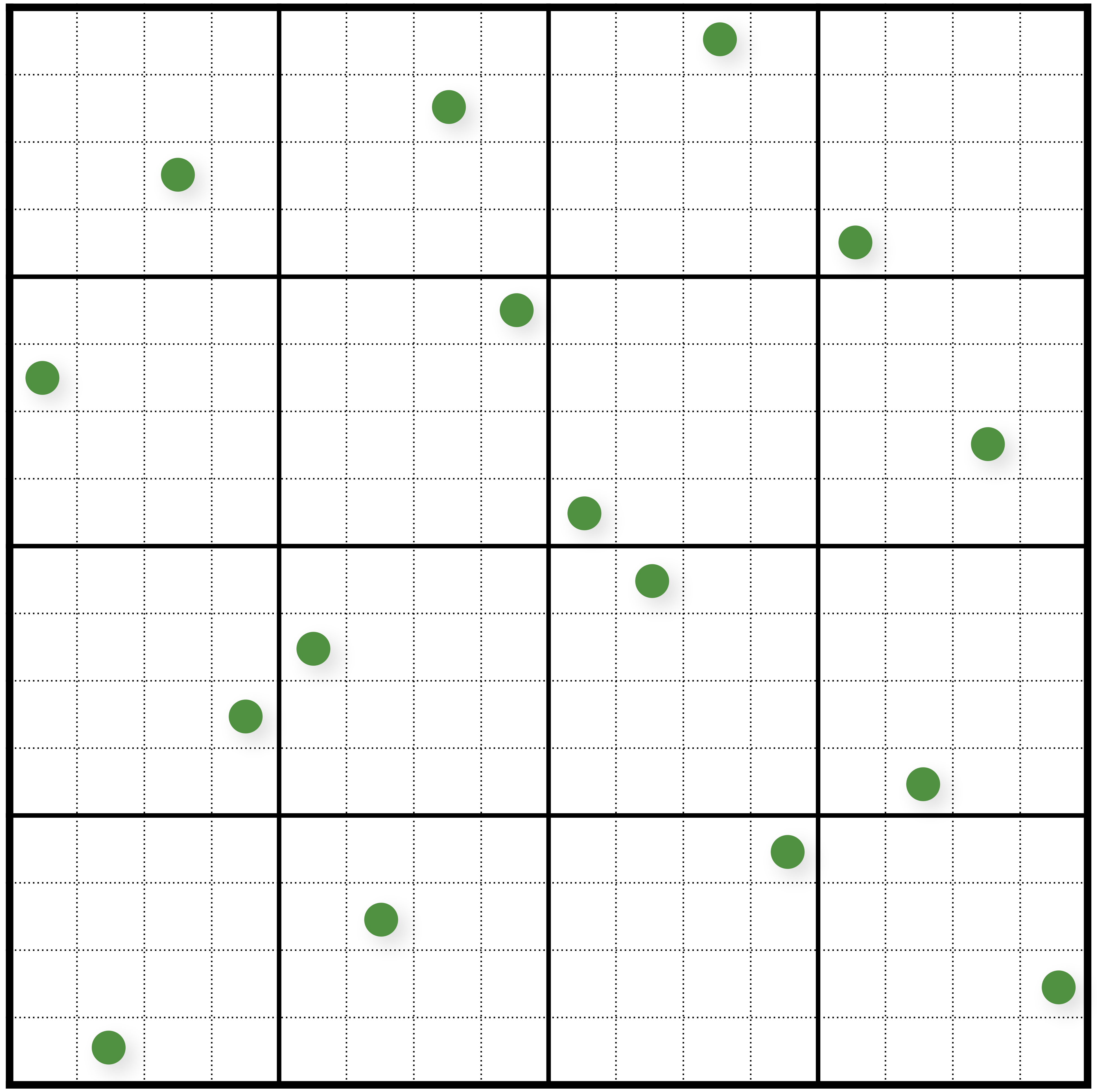
Shuffle y-coords

Multi-Jittered Sampling



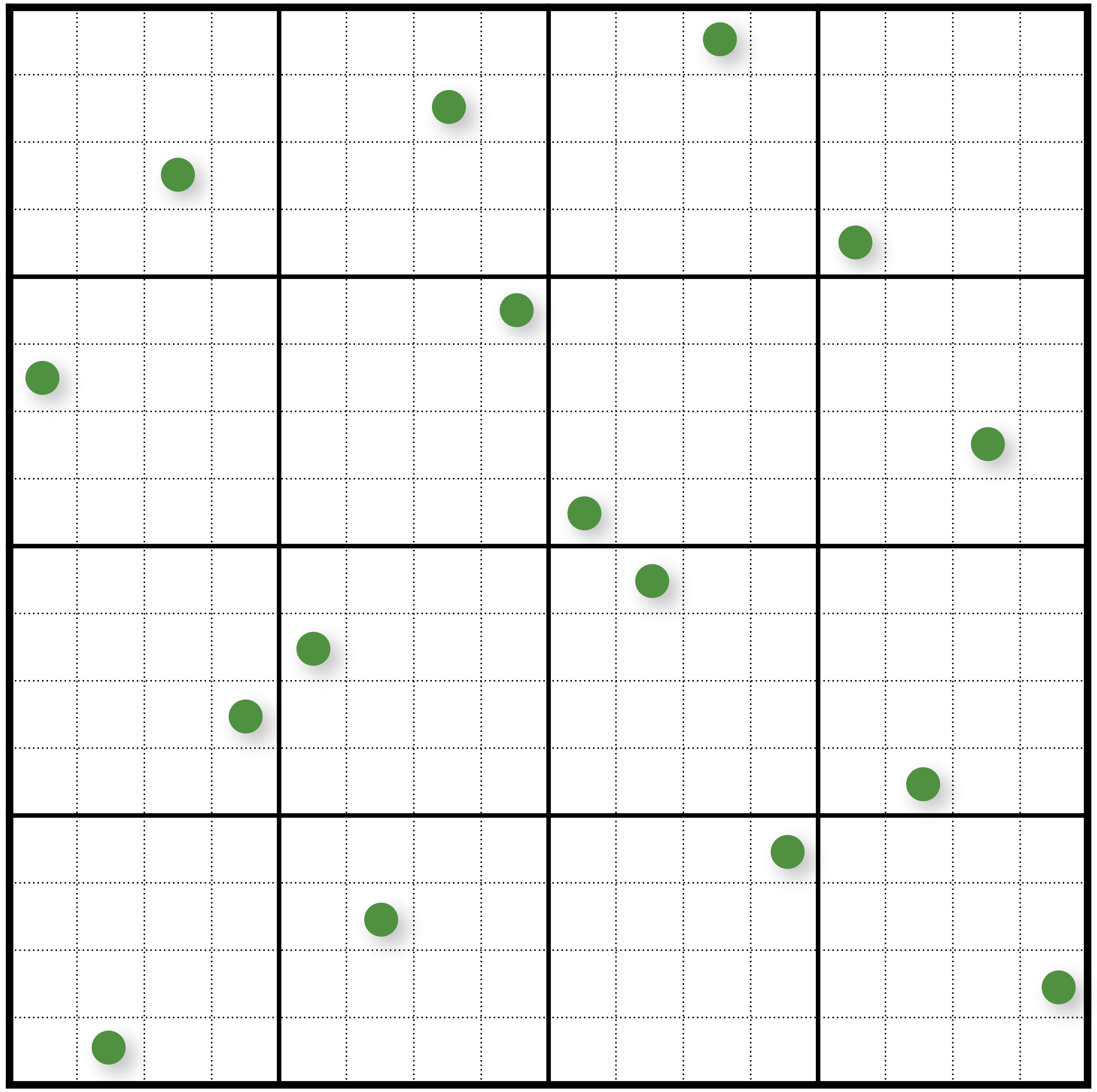
Shuffle y-coords

Multi-Jittered Sampling

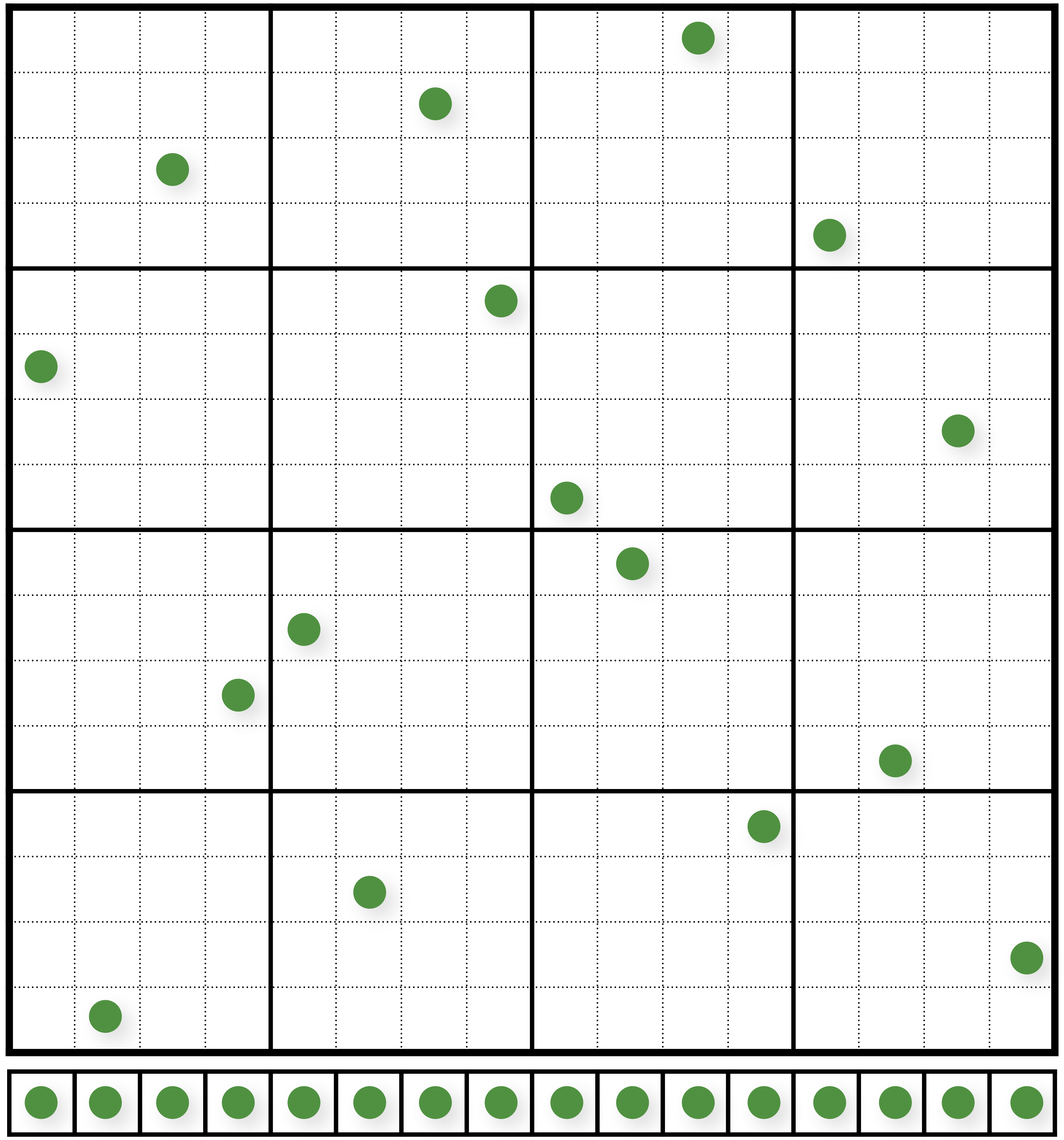


Shuffle y-coords

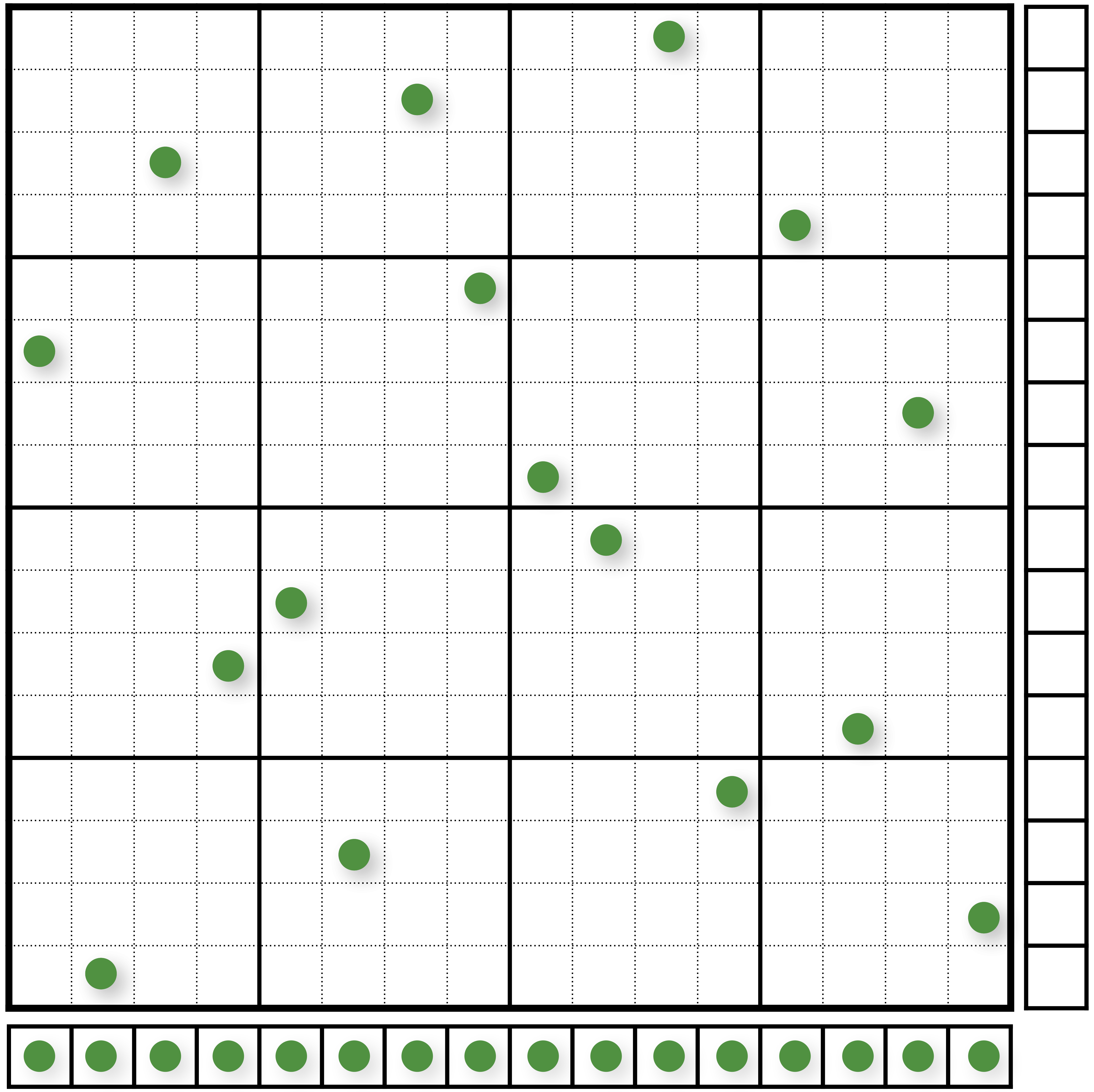
Multi-Jittered Sampling (Projections)



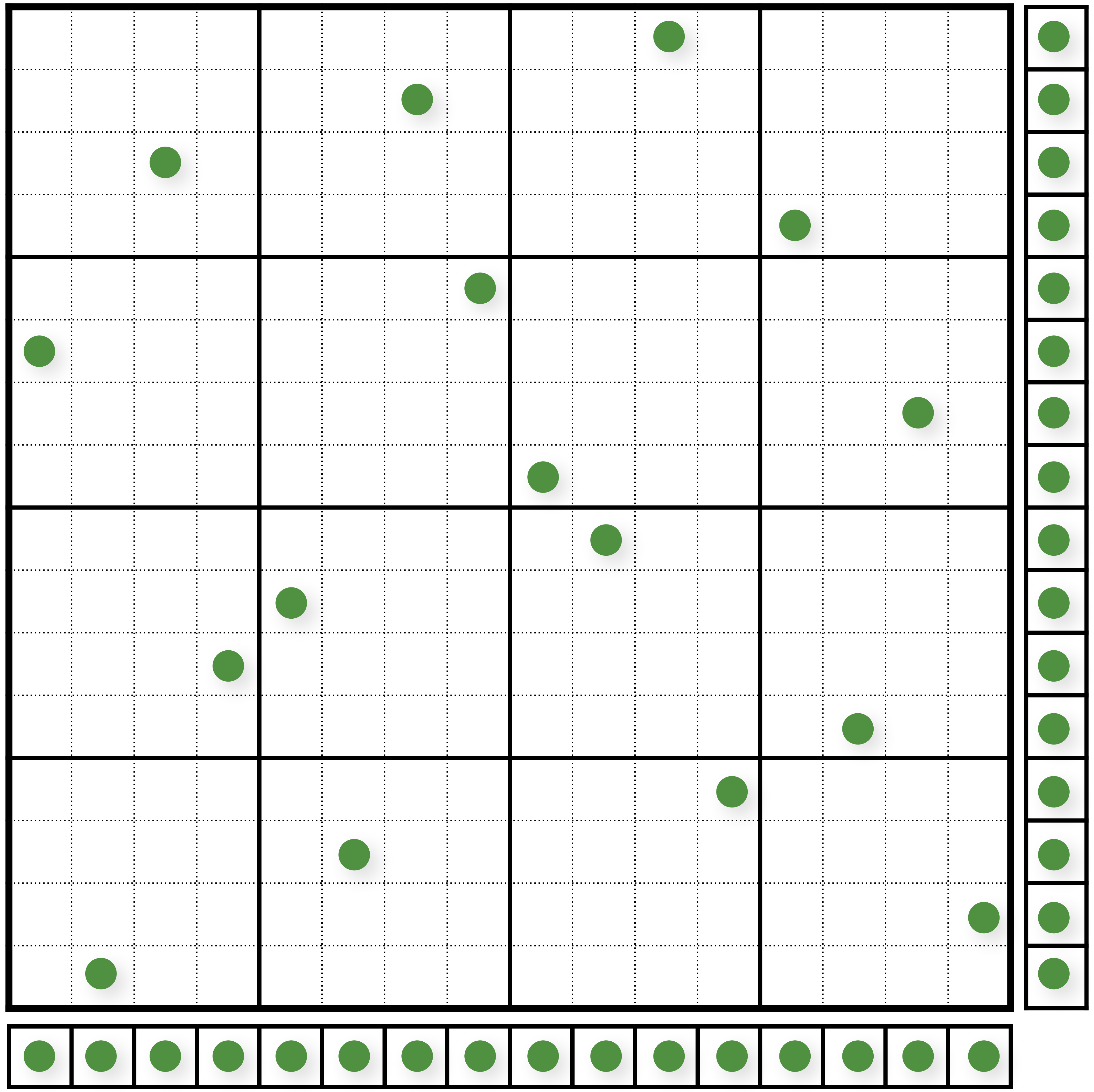
Multi-Jittered Sampling (Projections)



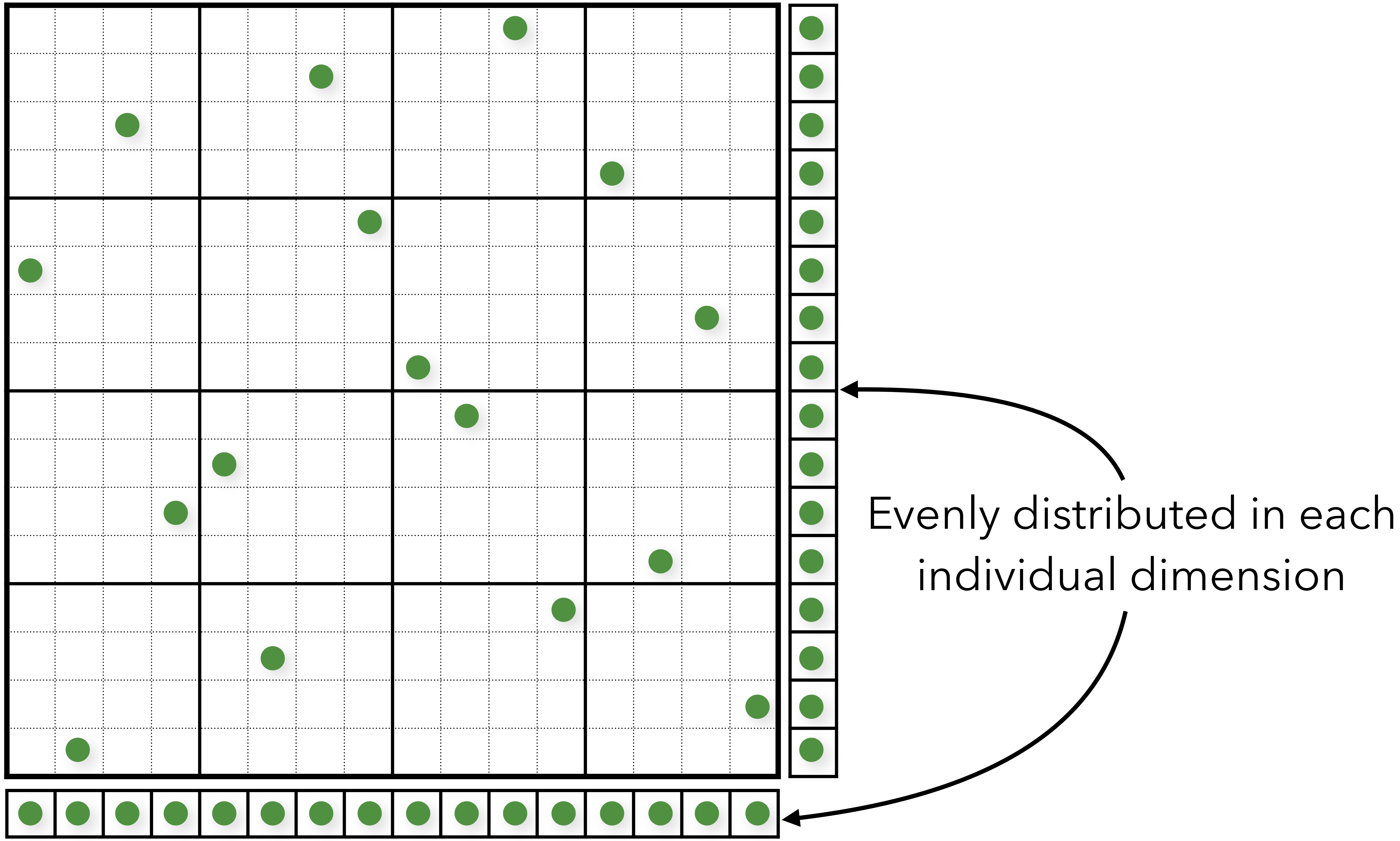
Multi-Jittered Sampling (Projections)



Multi-Jittered Sampling (Projections)

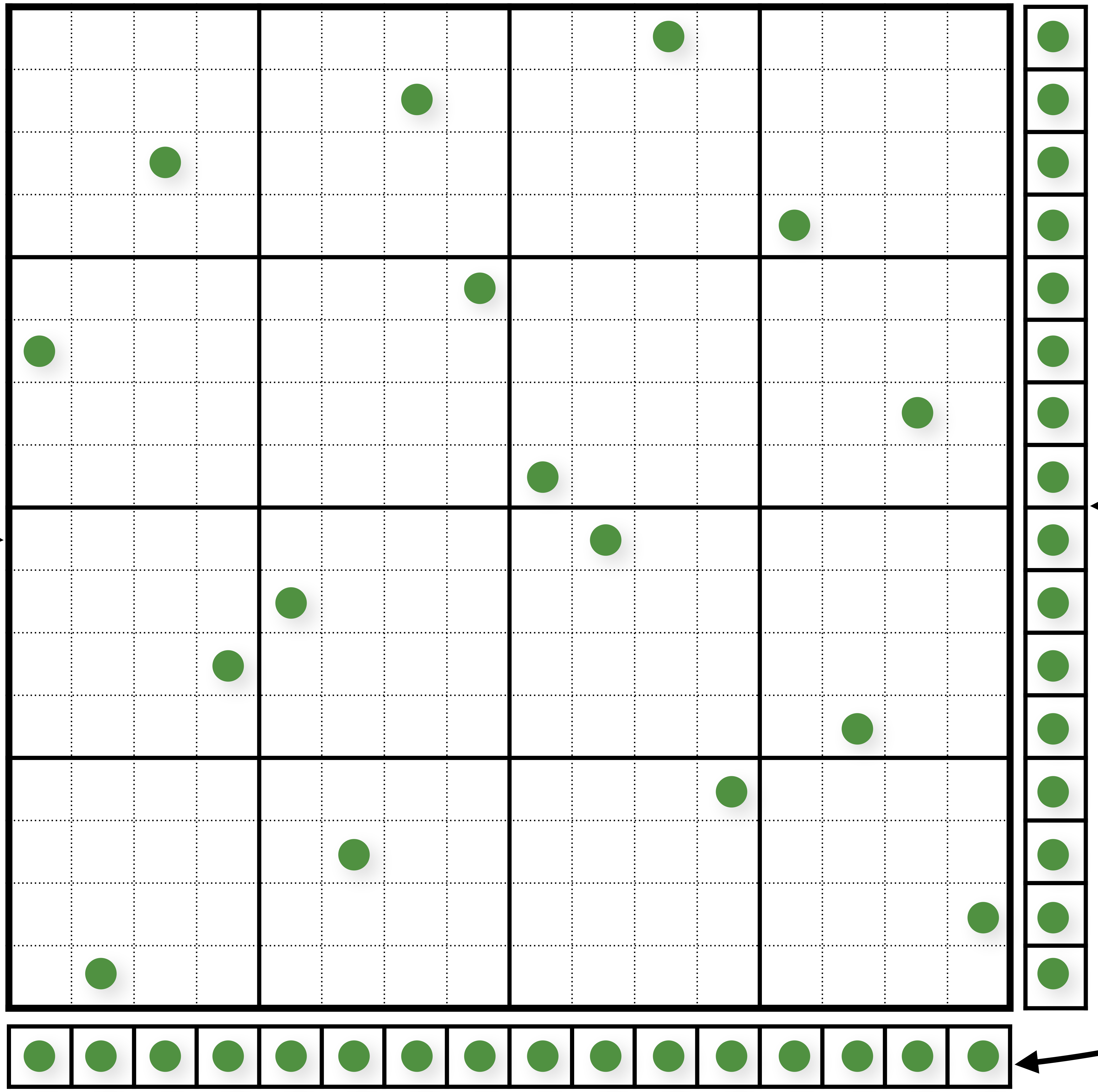


Multi-Jittered Sampling (Projections)



Multi-Jittered Sampling (Projections)

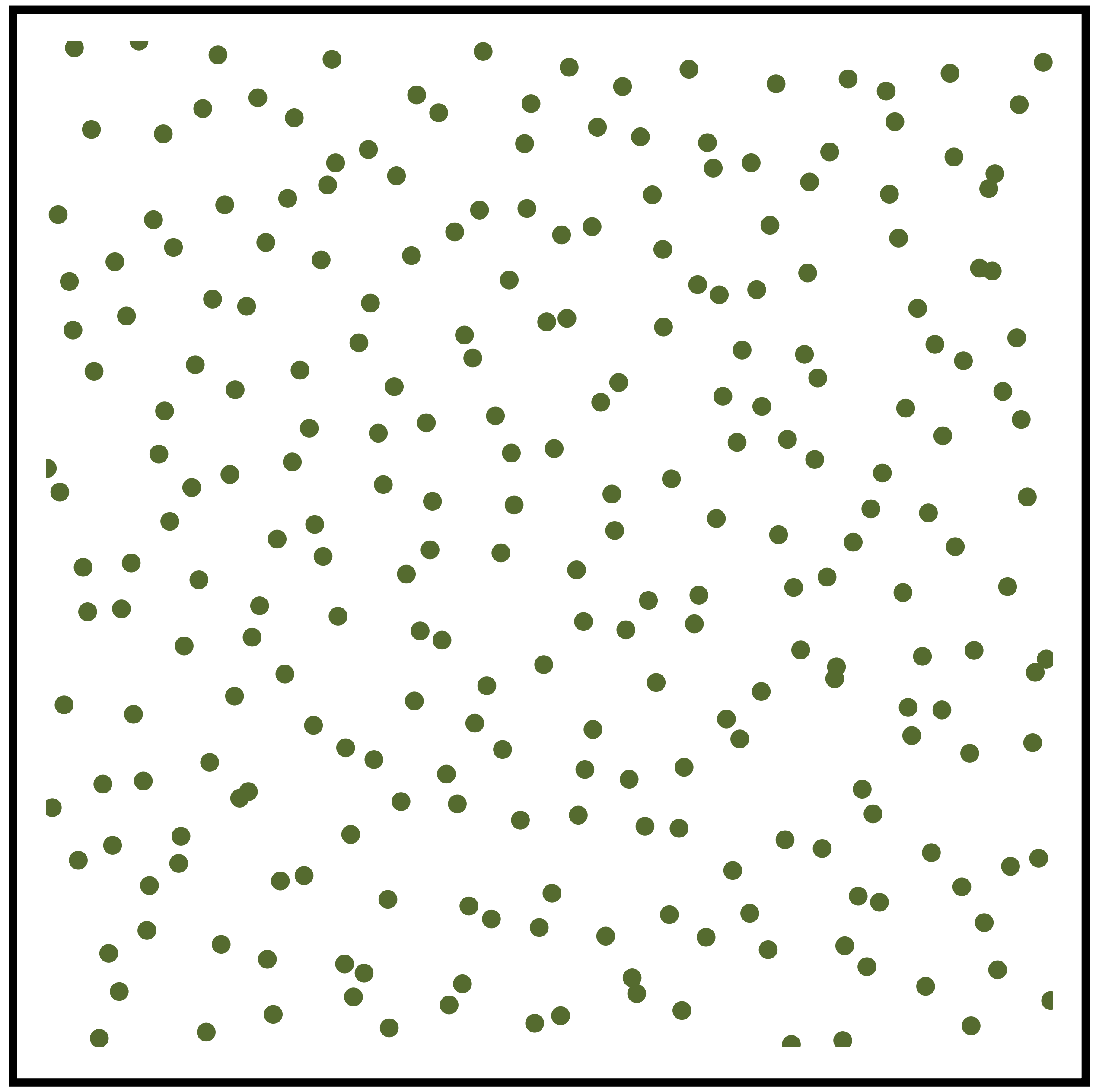
Evenly distributed in 2D!



Evenly distributed in each individual dimension

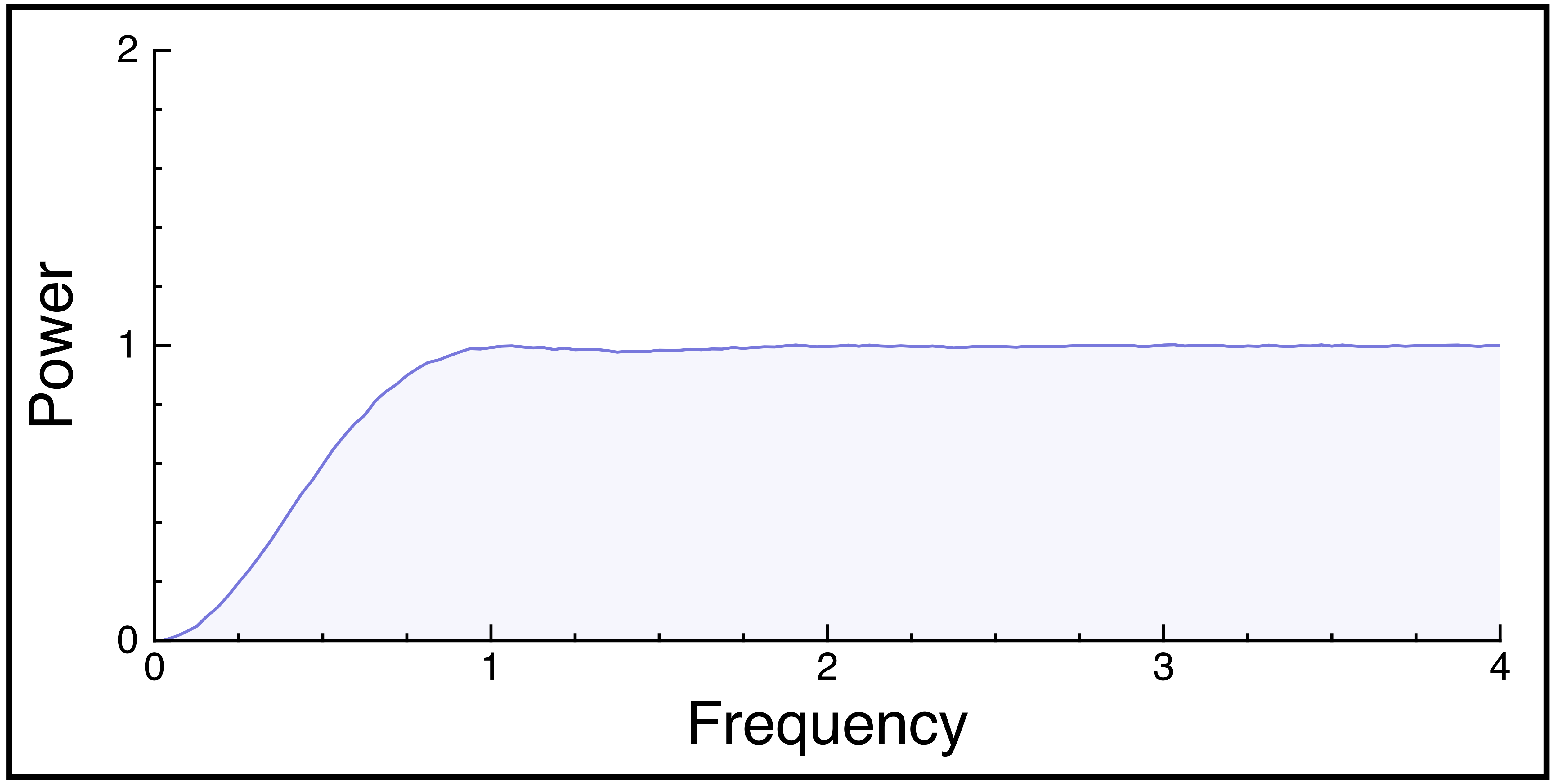
Multi-Jittered Sampling

Samples



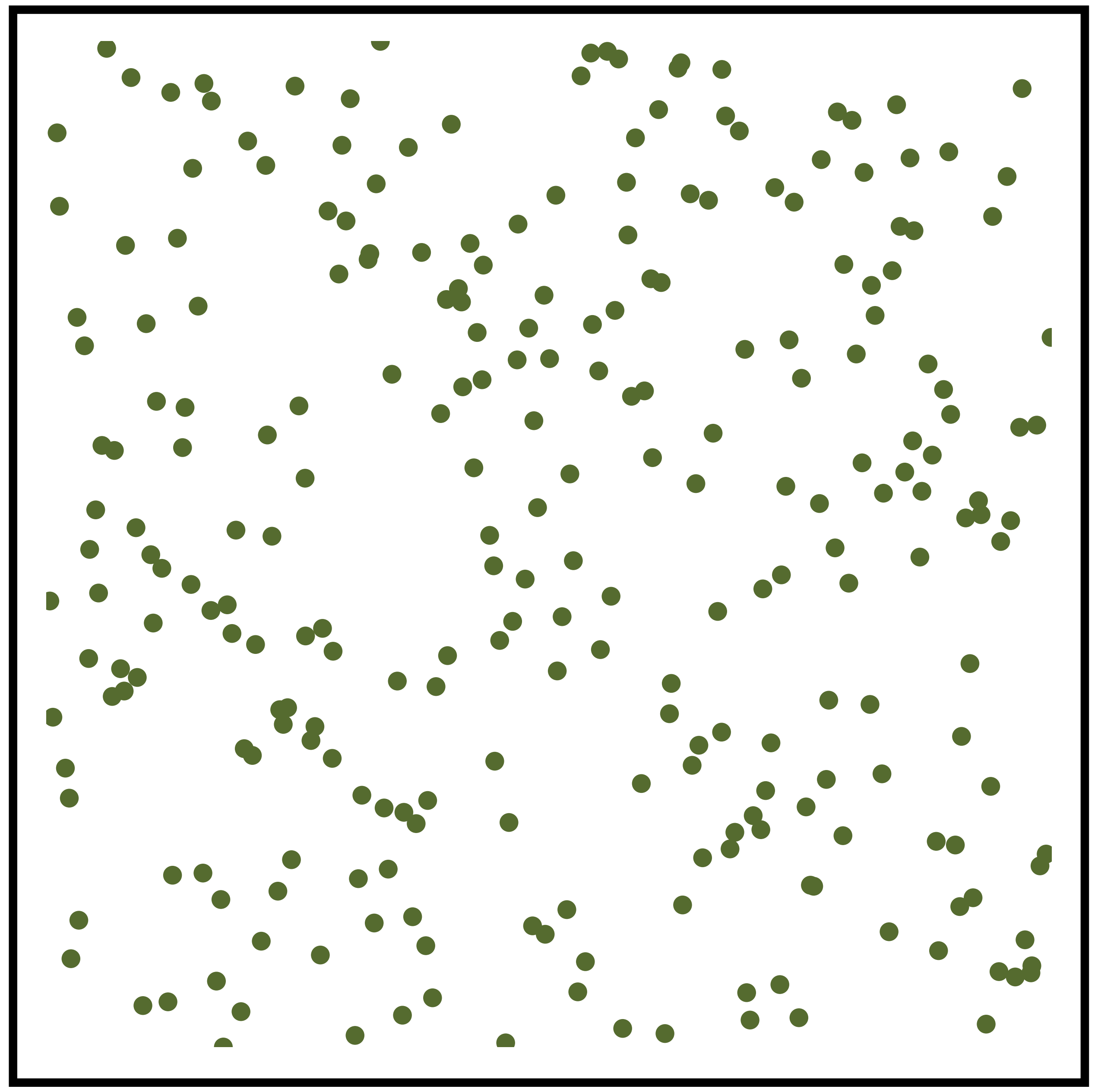
Expected power spectrum

Radial mean



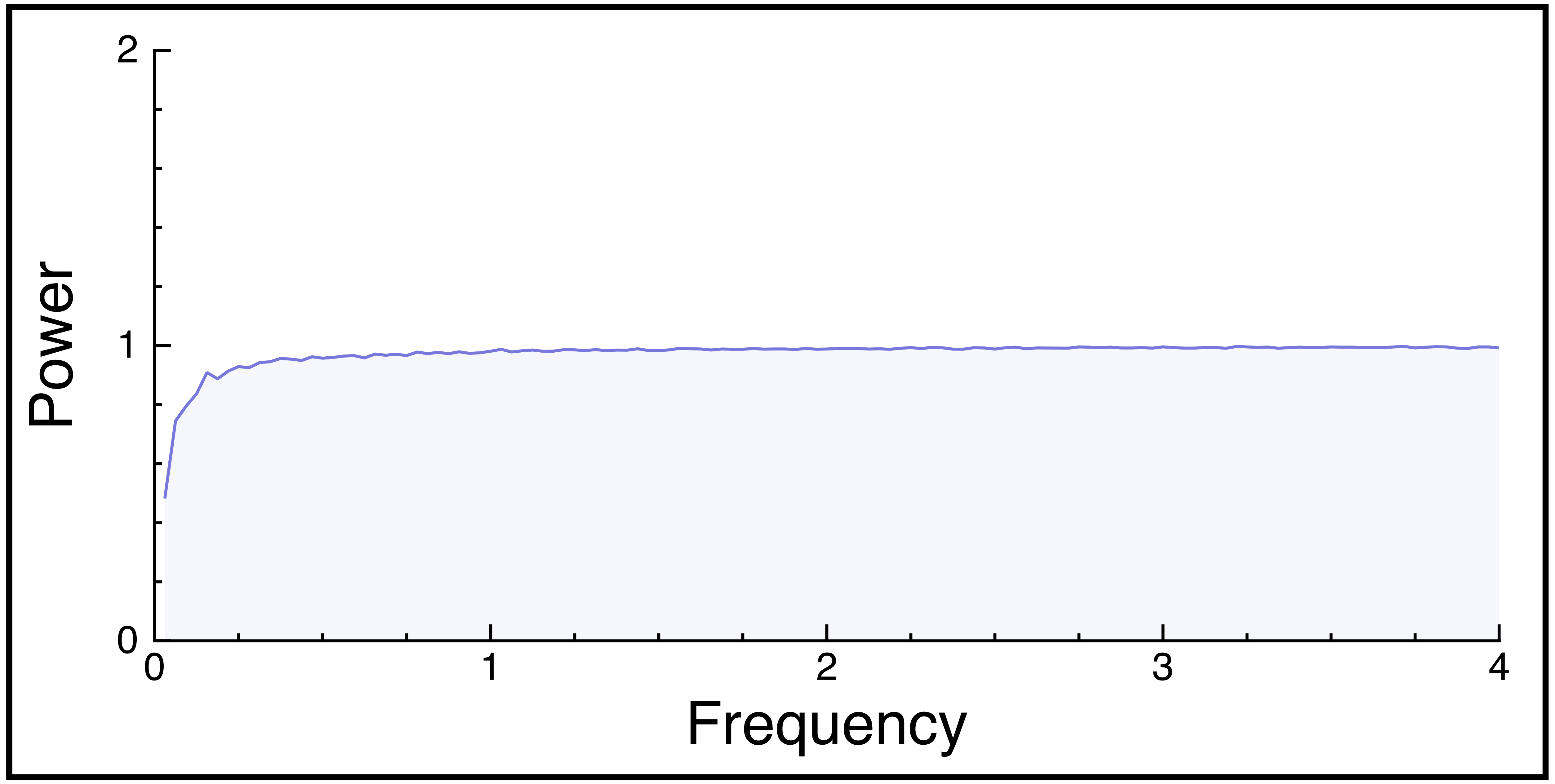
N-Rooks Sampling

Samples



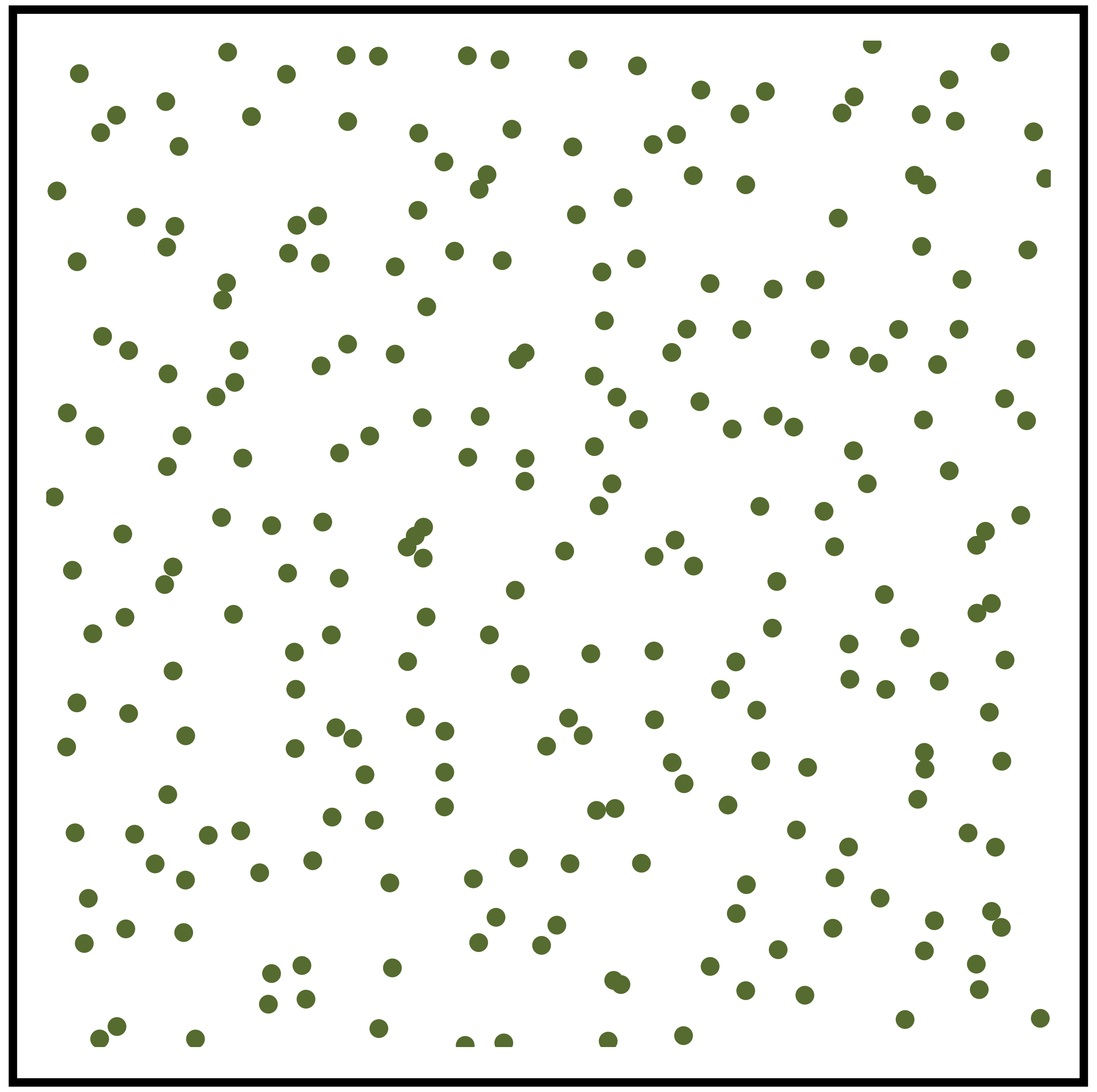
Expected power spectrum

Radial mean



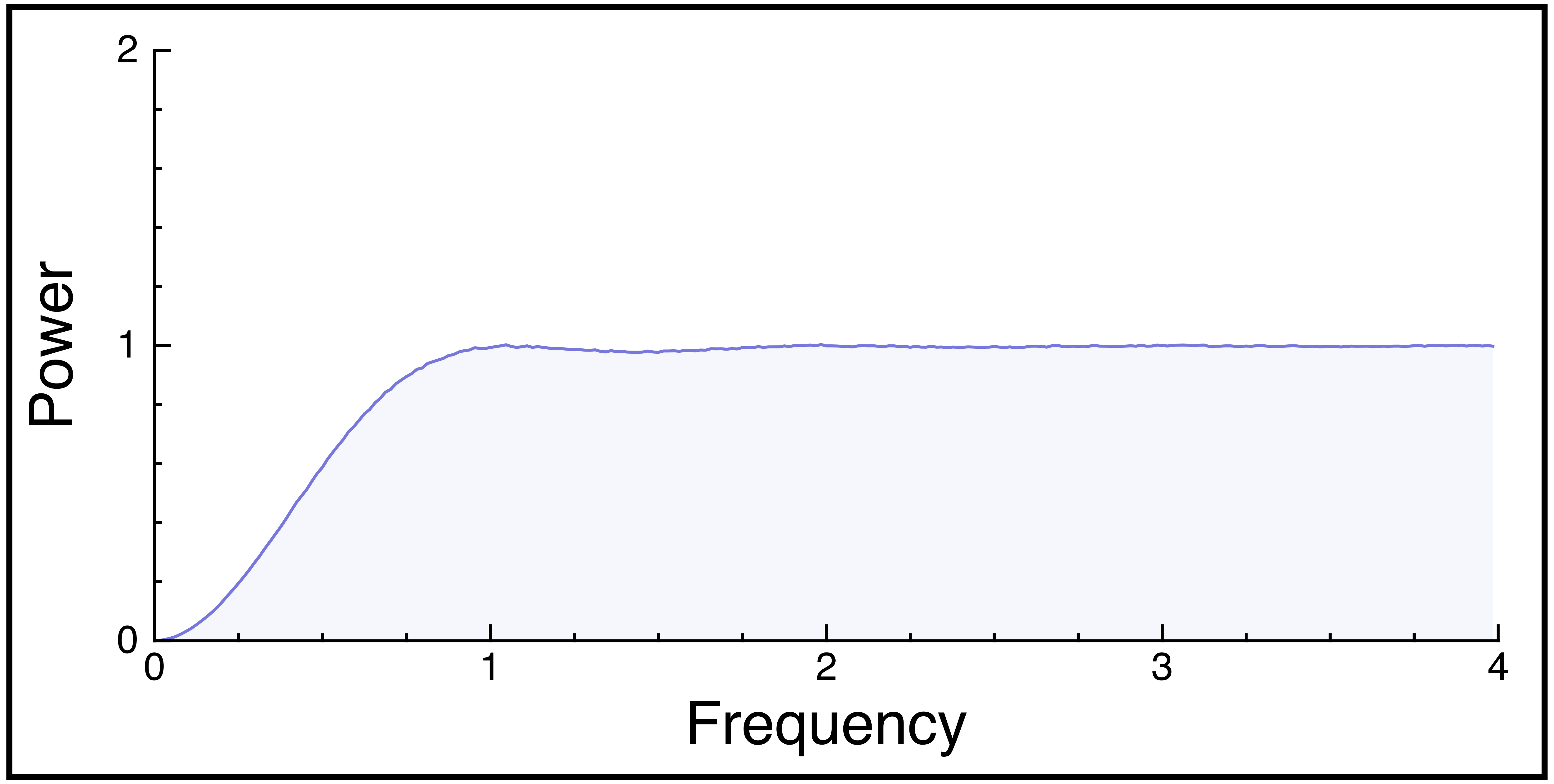
Jittered Sampling

Samples



Expected power spectrum

Radial mean



Poisson-Disk/Blue-Noise Sampling

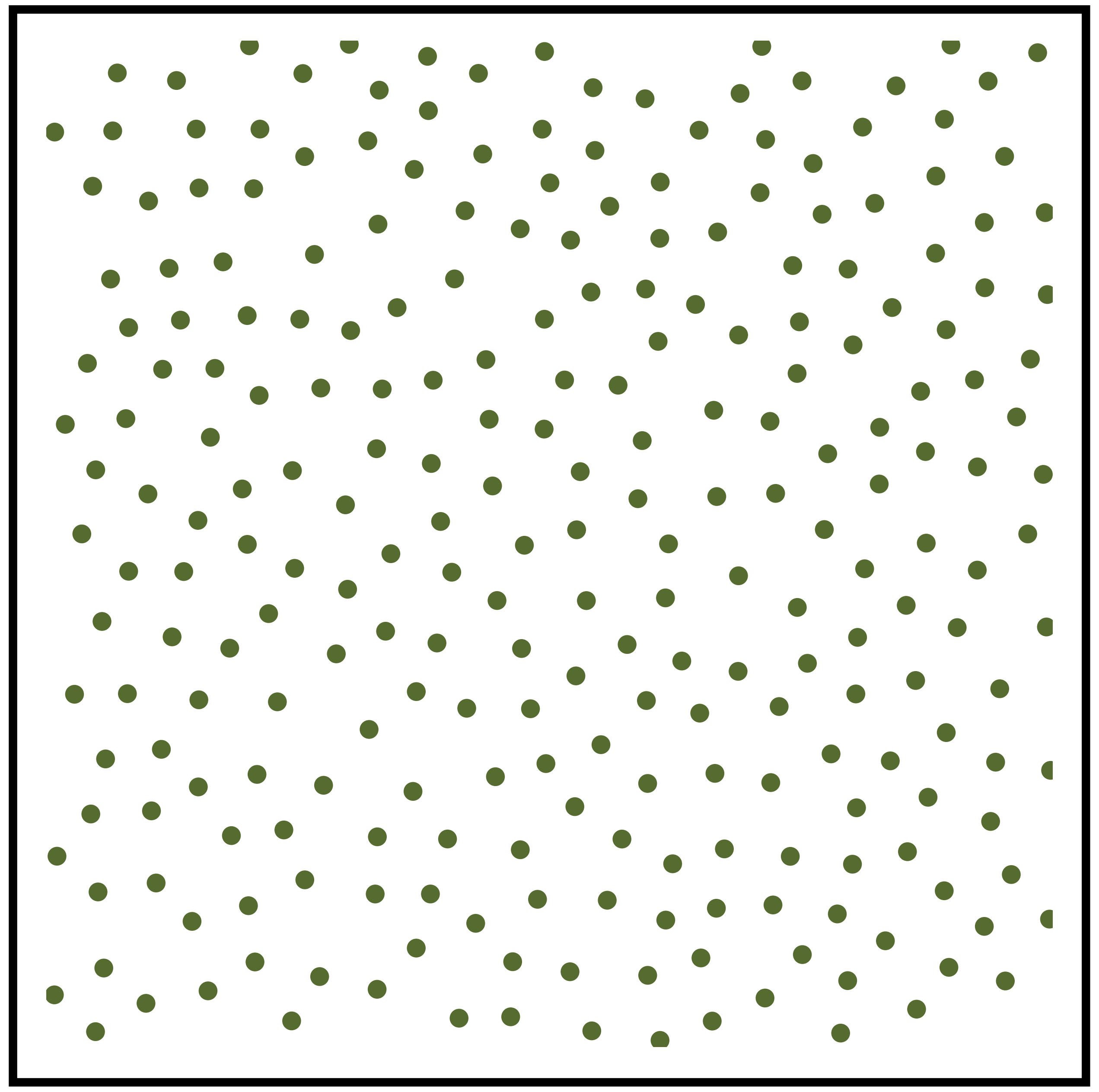
Enforce a minimum distance between points

Poisson-Disk Sampling:

- Mark A. Z. Dippé and Erling Henry Wold. “Antialiasing through stochastic sampling.” *ACM SIGGRAPH*, 1985.
- Robert L. Cook. “Stochastic sampling in computer graphics.” *ACM Transactions on Graphics*, 1986.
- Ares Lagae and Philip Dutré. “A comparison of methods for generating Poisson disk distributions.” *Computer Graphics Forum*, 2008.

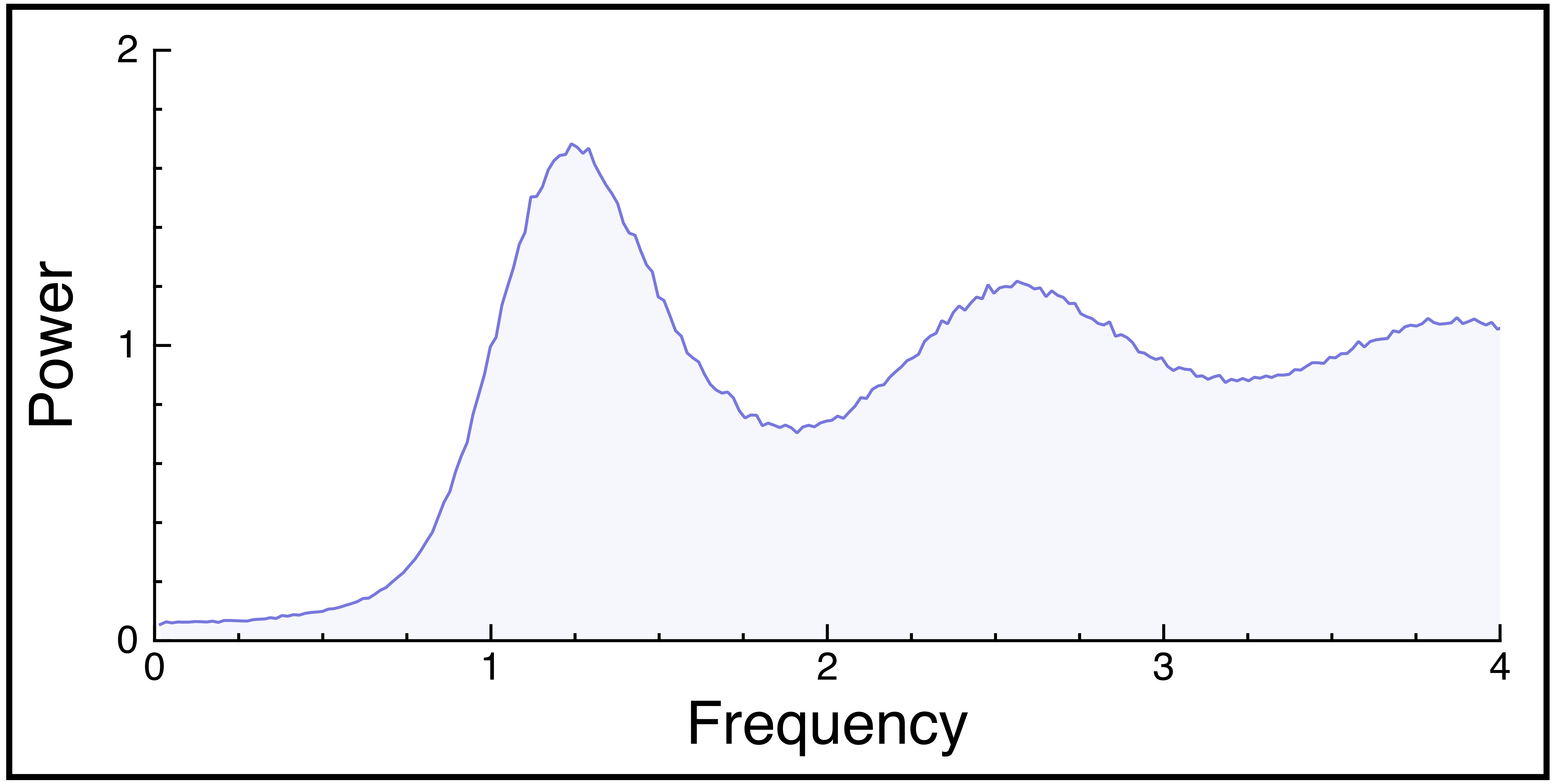
Poisson Disk Sampling

Samples



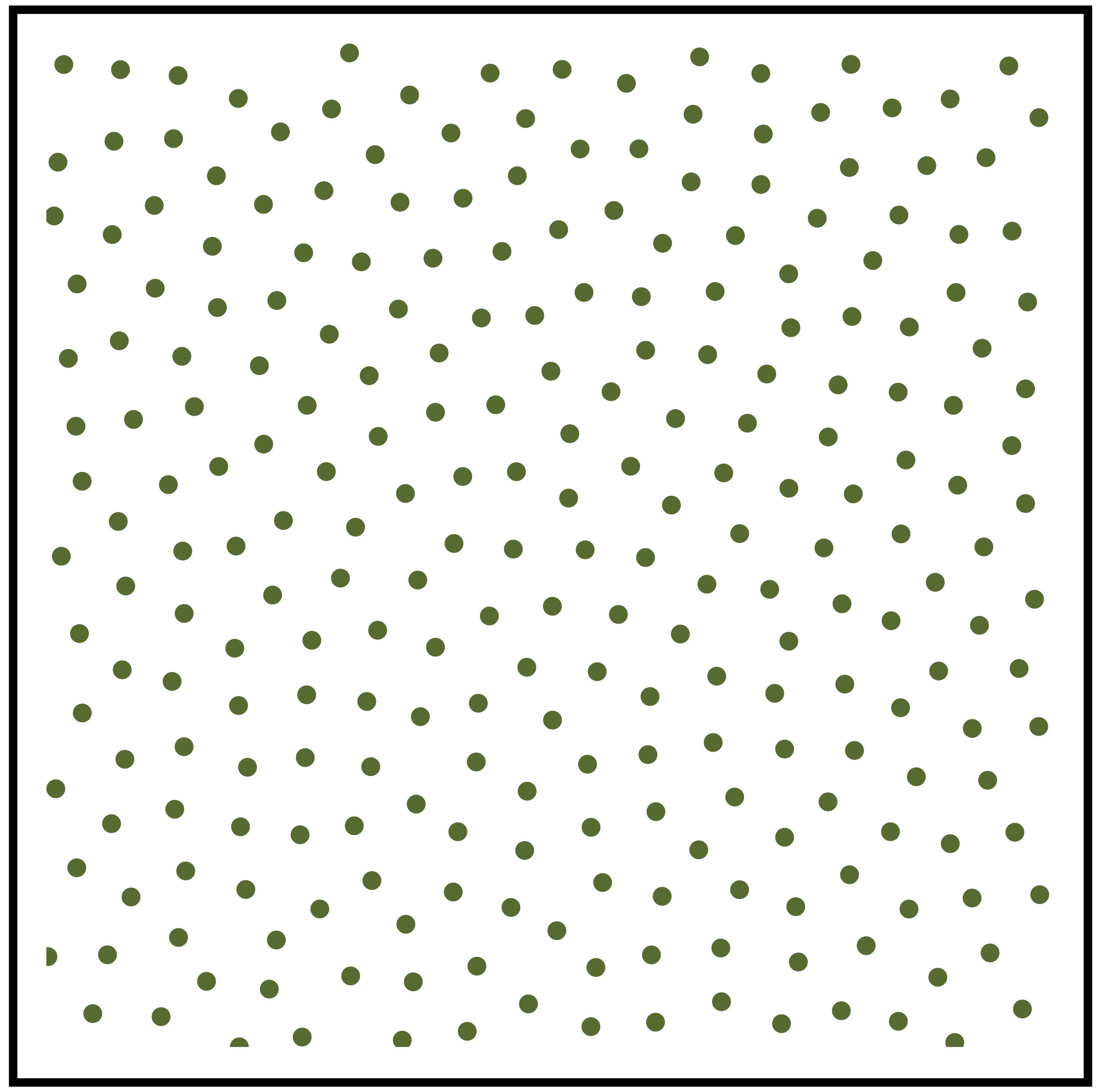
Expected power spectrum

Radial mean



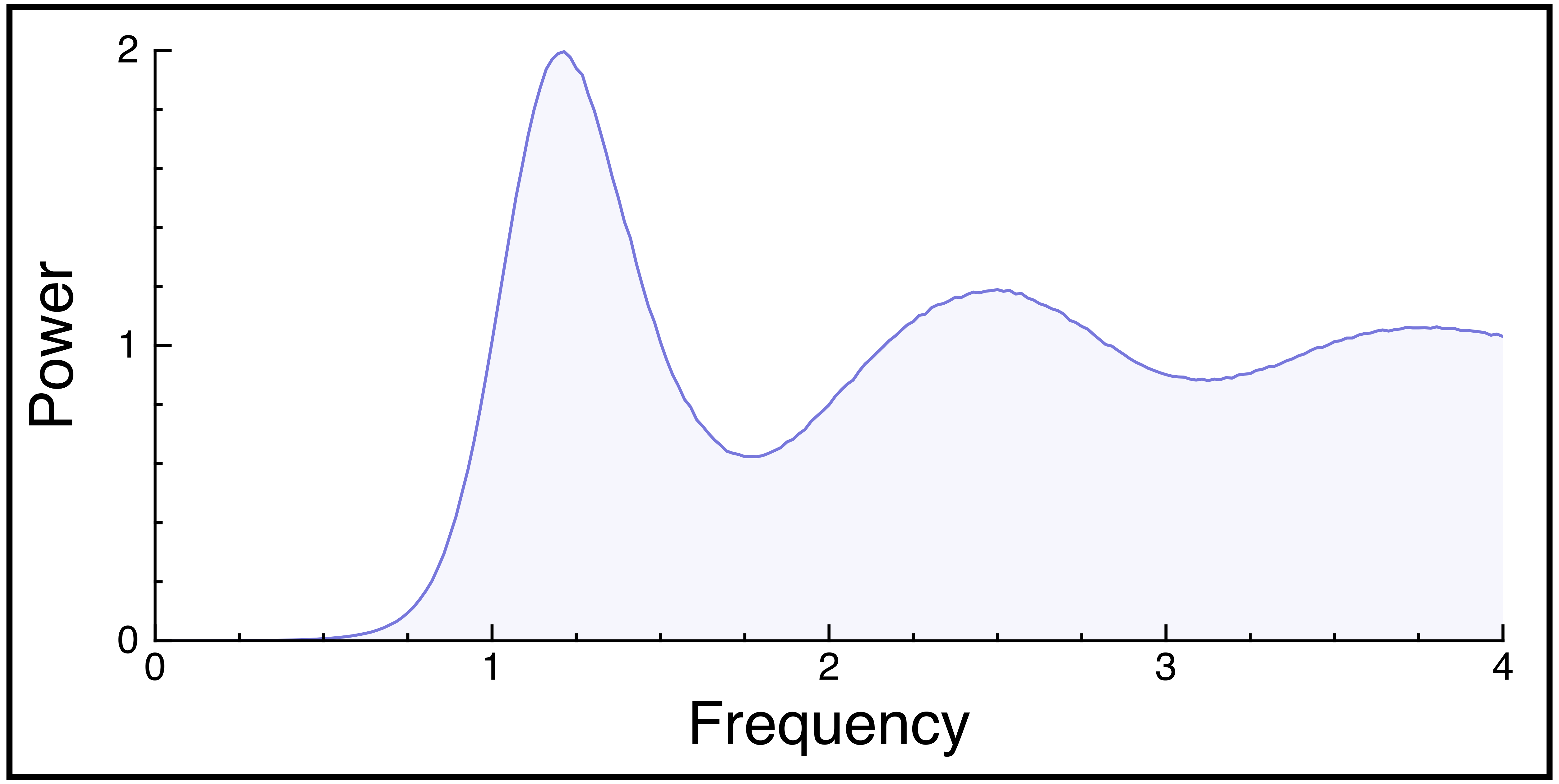
CCVT Sampling [Balzer et al. 2009]

Samples



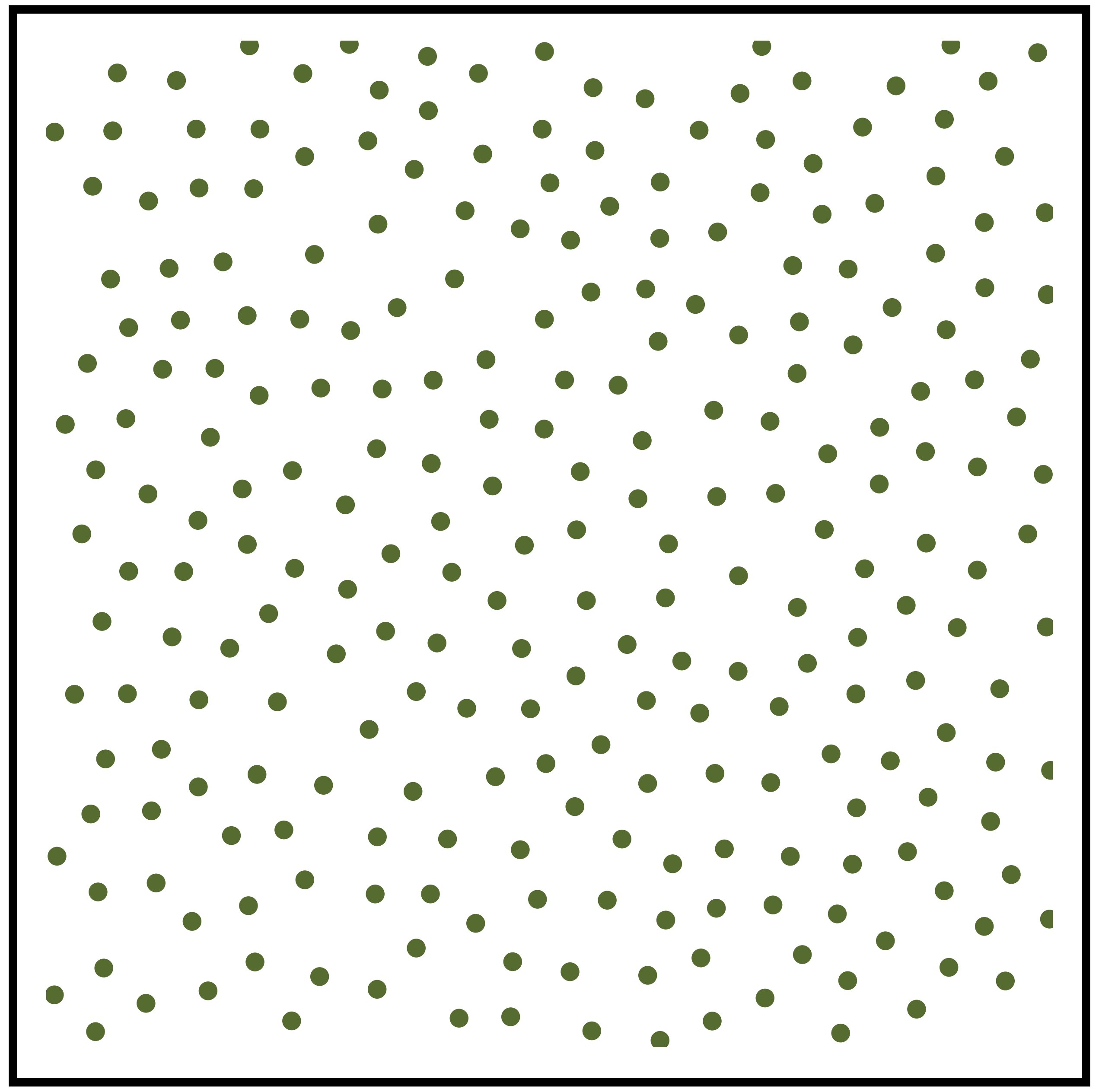
Expected power spectrum

Radial mean



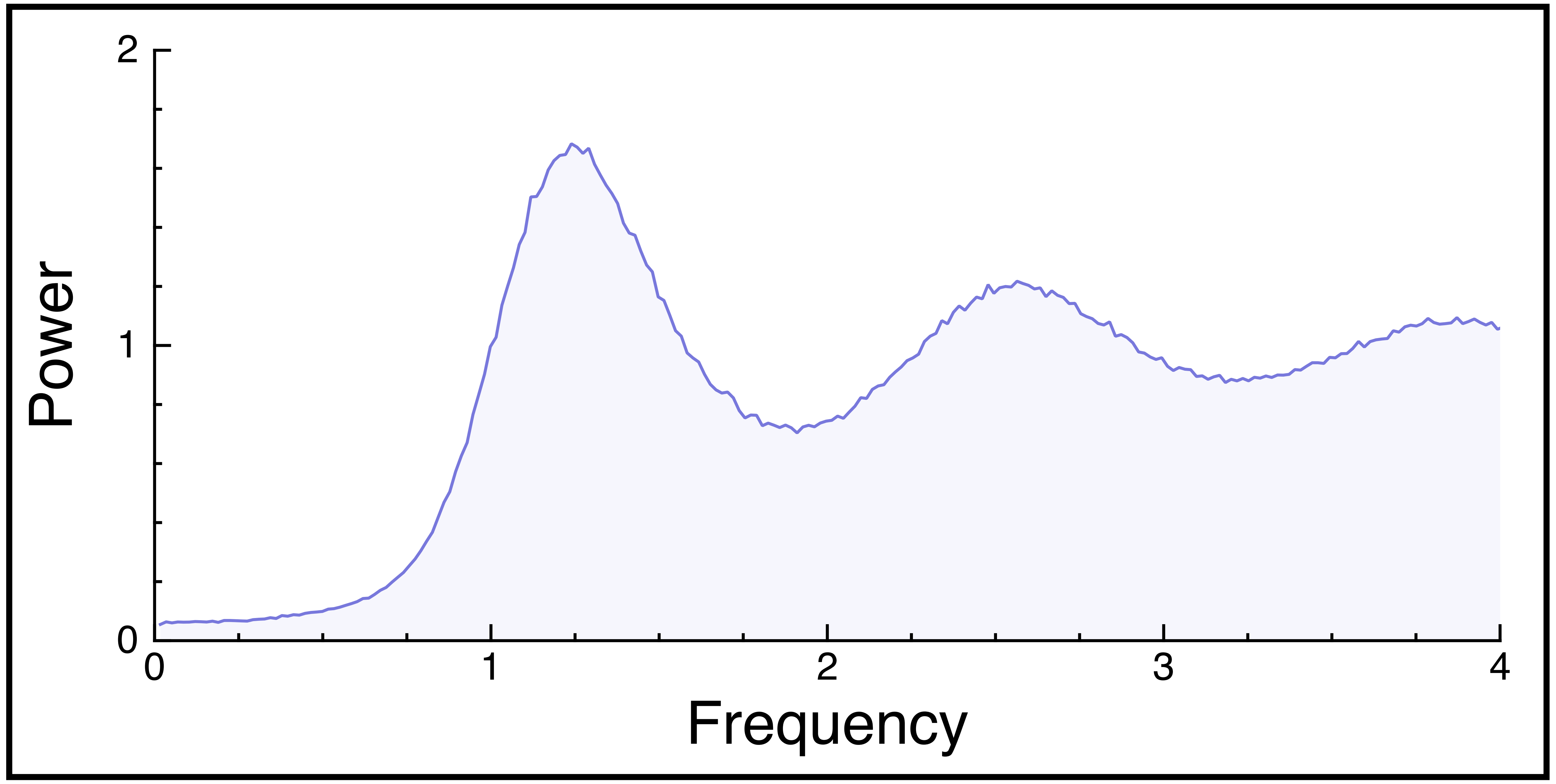
Poisson Disk Sampling

Samples



Expected power spectrum

Radial mean



Low-Discrepancy Sampling

Deterministic sets of points specially crafted to be evenly distributed (have low discrepancy).

Entire field of study called Quasi-Monte Carlo (QMC)

The Van der Corput Sequence

Radical Inverse Φ_b in base 2

Subsequent points "fall into biggest holes"

k	Base 2	Φ_b

The Van der Corput Sequence

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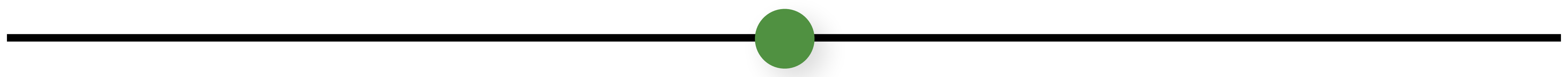
k	Base 2	Φ_b
1	1	.1 = 1/2

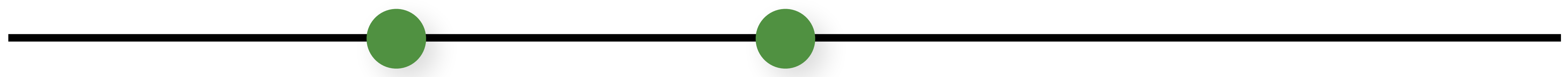


The Van der Corput Sequence

Radical Inverse Φ_b in base 2

Subsequent points “fall into biggest holes”

k	Base 2	Φ_b
1	1	.1 = 1/2
2	10	.01 = 1/4



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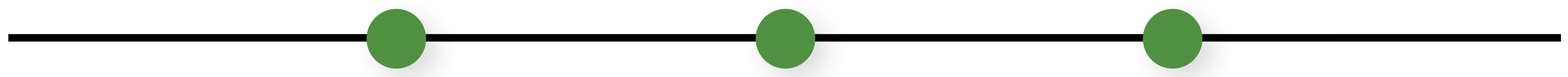
Subsequent points “fall into biggest holes”

k	Base 2	Φ_b
1	1	.1 = 1/2
2	10	.01 = 1/4
3	11	.11 = 3/4



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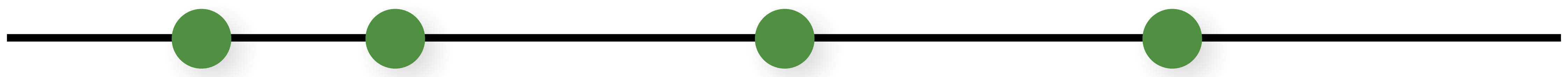
Subsequent points “fall into biggest holes”

k	Base 2	Φ_b
1	1	.1 = 1/2
2	10	.01 = 1/4
3	11	.11 = 3/4
4	100	.001 = 1/8



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1	1	.1 = 1/2
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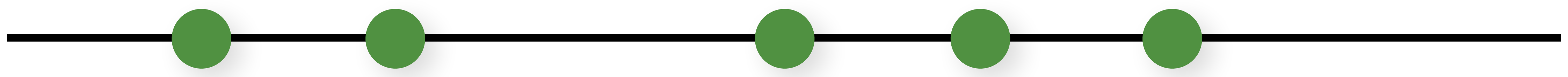
3	11	.11 = 3/4
4	100	.001 = 1/8
5	101	.101 = 5/8



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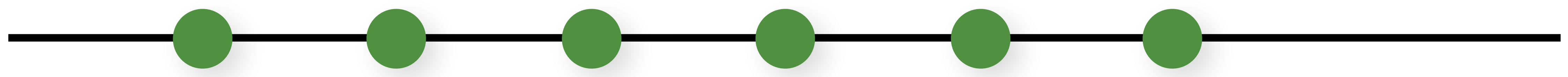
Subsequent points “fall into biggest holes”

k	Base 2	Φ_b
1	1	.1 = 1/2
2	10	.01 = 1/4
3	11	.11 = 3/4
4	100	.001 = 1/8
5	101	.101 = 5/8
6	110	.011 = 3/8



The Van der Corput Sequence

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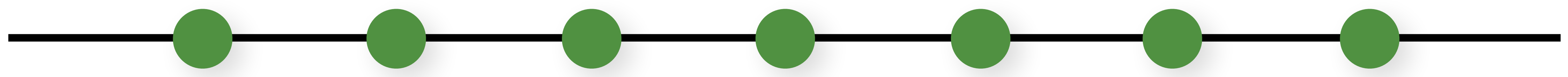
Subsequent points “fall into biggest holes”

k	Base 2	Φ_b
1	1	.1 = 1/2
2	10	.01 = 1/4
3	11	.11 = 3/4
4	100	.001 = 1/8
5	101	.101 = 5/8
6	110	.011 = 3/8
7	111	.111 = 7/8



The Van der Corput Sequence

Radical Inverse Φ_b in base 2

Subsequent points “fall into biggest holes”

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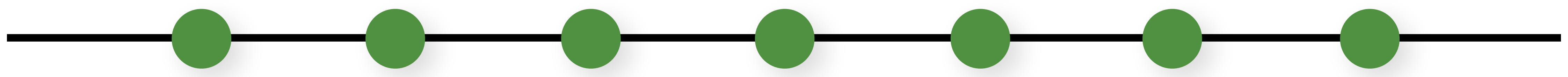
1	1	.1 = 1/2
2	10	.01 = 1/4
3	11	.11 = 3/4
4	100	.001 = 1/8
5	101	.101 = 5/8
6	110	.011 = 3/8
7	111	.111 = 7/8
...		



Halton and Hammersley Points

Halton: Radical inverse with different base for each dimension:

$$\vec{x}_k = (\Phi_2(k), \Phi_3(k), \Phi_5(k), \ldots, \Phi_{p_n}(k))$$

Halton and Hammersley Points

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$$\vec{x}_k = (\Phi_2(k), \Phi_3(k), \Phi_5(k), \ldots, \Phi_{p_n}(k))$$

- The bases should all be relatively prime.

Halton and Hammersley Points

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- Incremental/progressive generation of samples

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- The bases should all be relatively prime.
- Incremental/progressive generation of samples

Hammersley: Same as Halton, but first dimension is k/N:

$$\vec{x}_k = (k/N, \Phi_2(k), \Phi_3(k), \Phi_5(k), \ldots, \Phi_{p_n}(k))$$

Halton and Hammersley Points

Halton: Radical inverse with different base for each dimension:

$$\vec{x}_k = (\Phi_2(k), \Phi_3(k), \Phi_5(k), \ \ldots \ , \Phi_{p_n}(k))$$

- The bases should all be relatively prime.

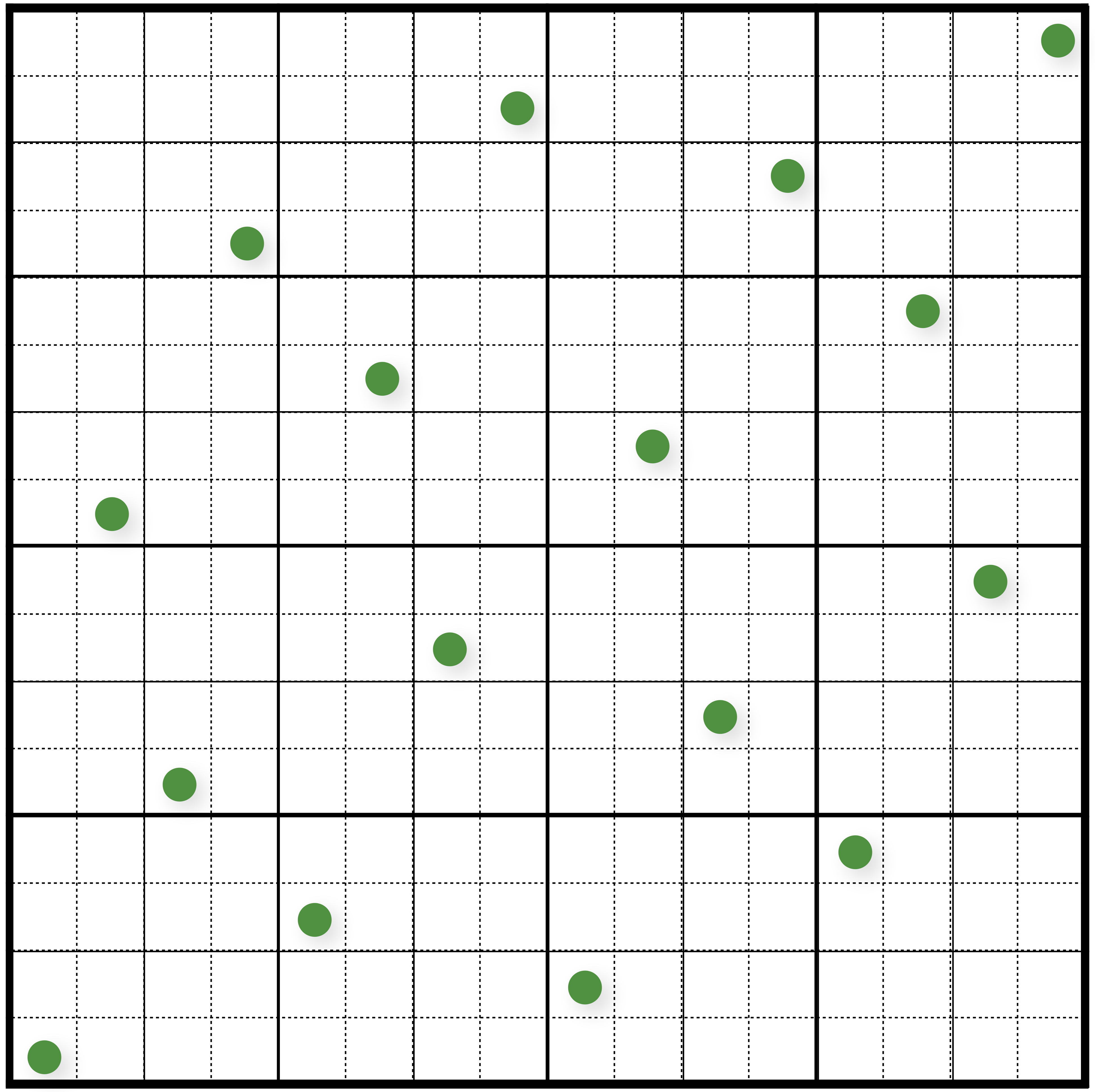
- Incremental/progressive generation of samples

Hammersley: Same as Halton, but first dimension is k/N:

$$\vec{x}_k = ({}^{k}/_{N}, \Phi_2(k), \Phi_3(k), \Phi_5(k), \ \ldots \ , \Phi_{p_n}(k))$$

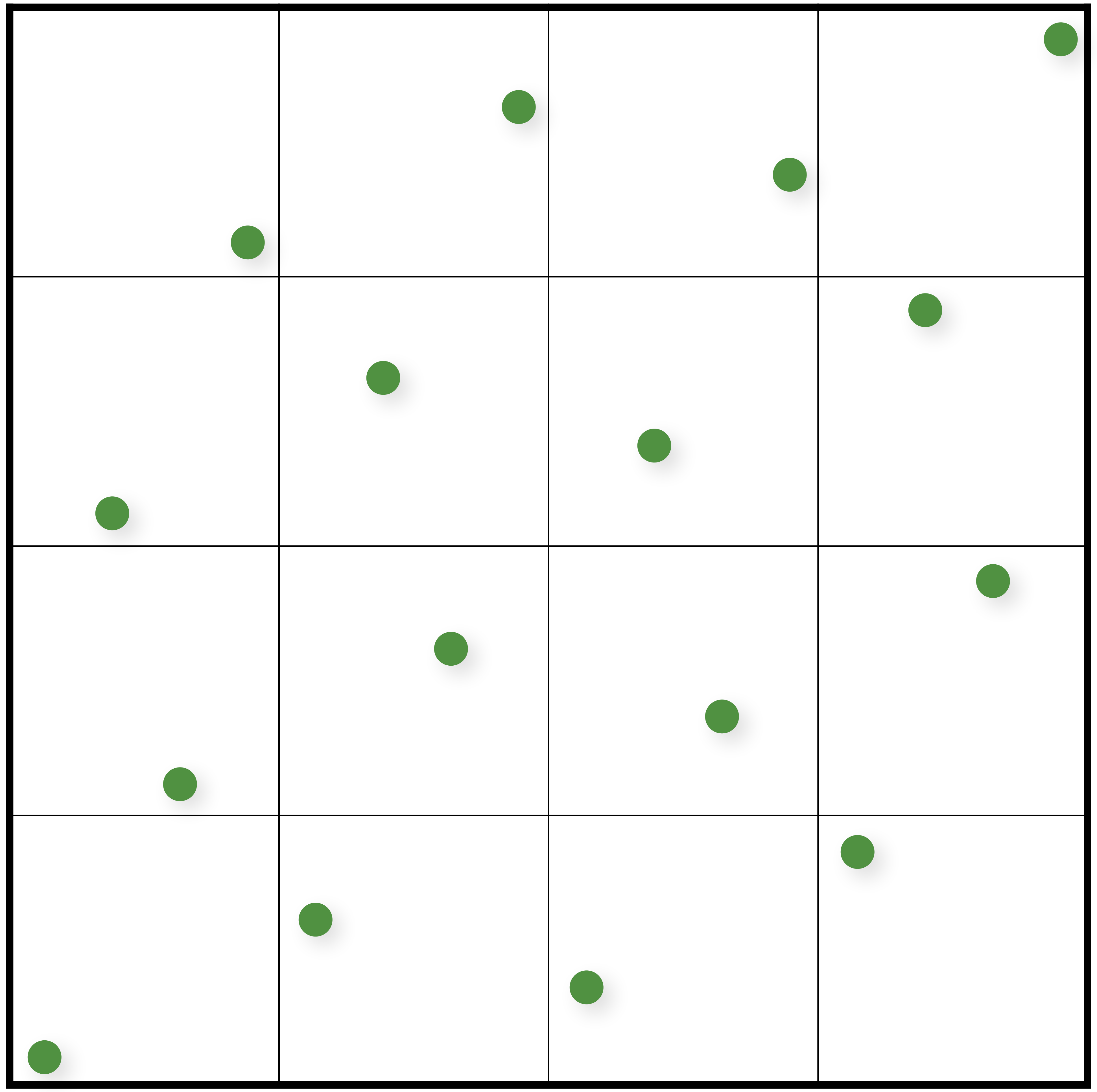
- Not incremental, need to know sample count, N, in advance

The Hammersley Sequence



1 sample in each “elementary interval”

The Hammersley Sequence



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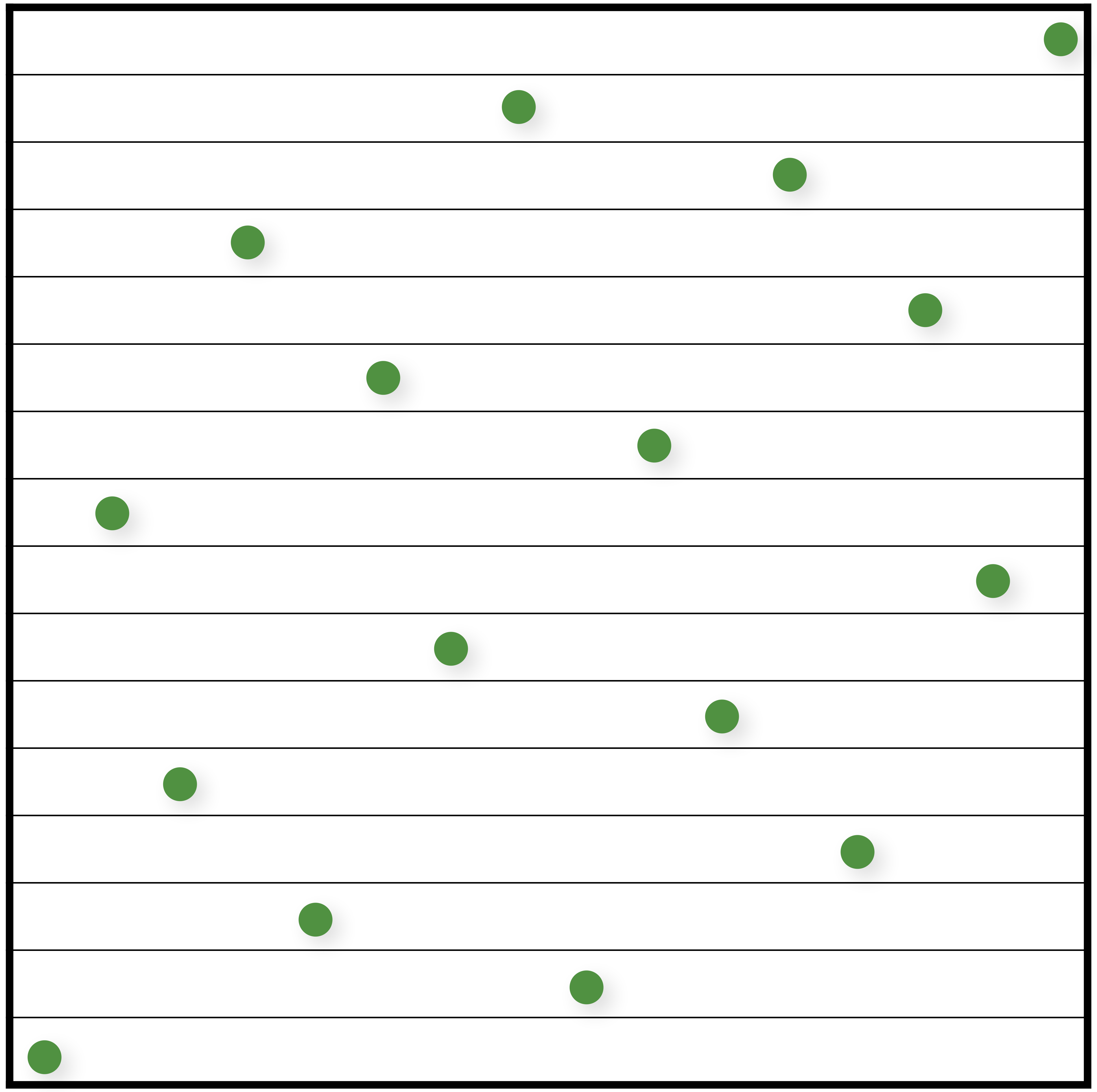
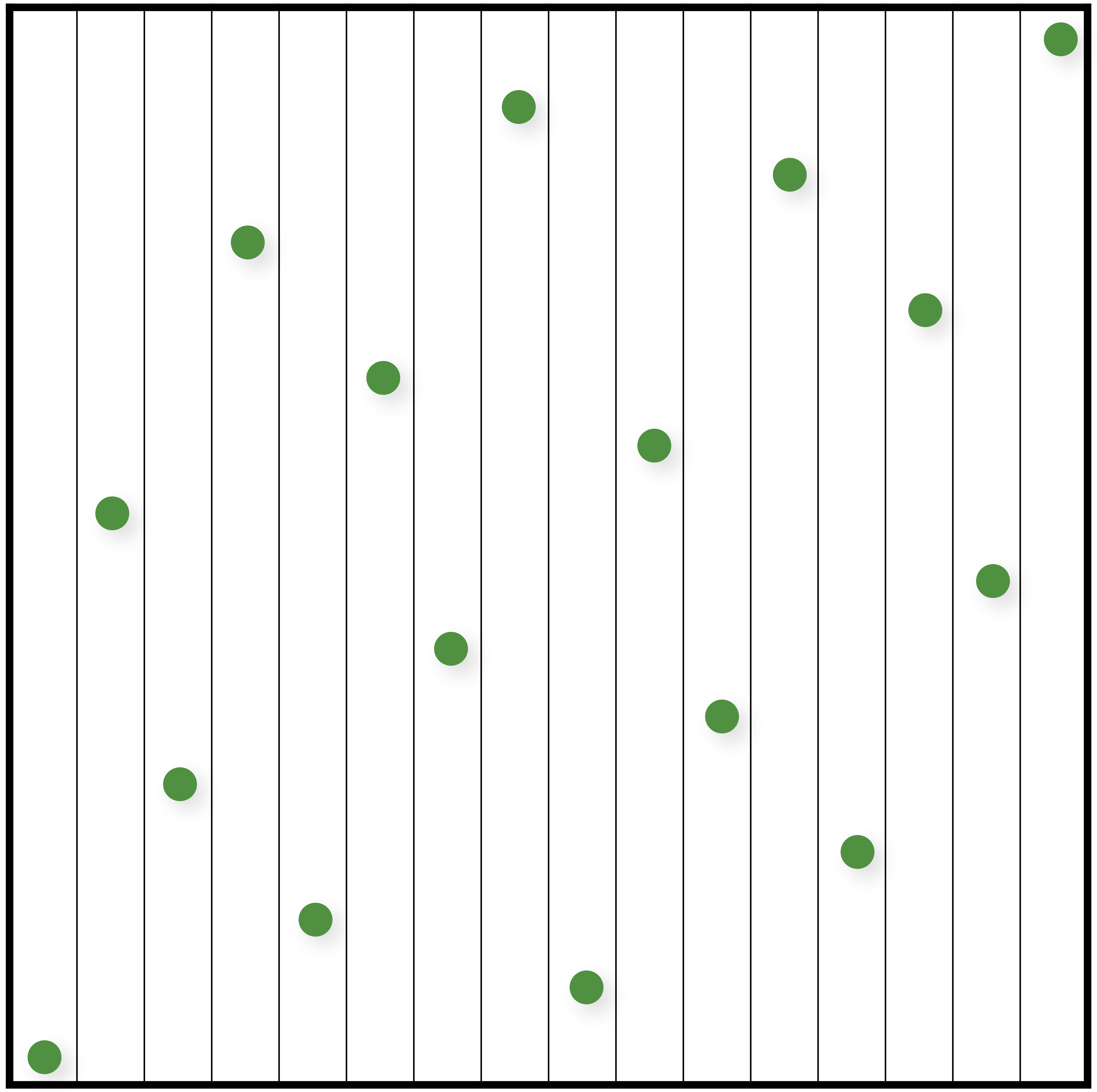
The Hammersley Sequence



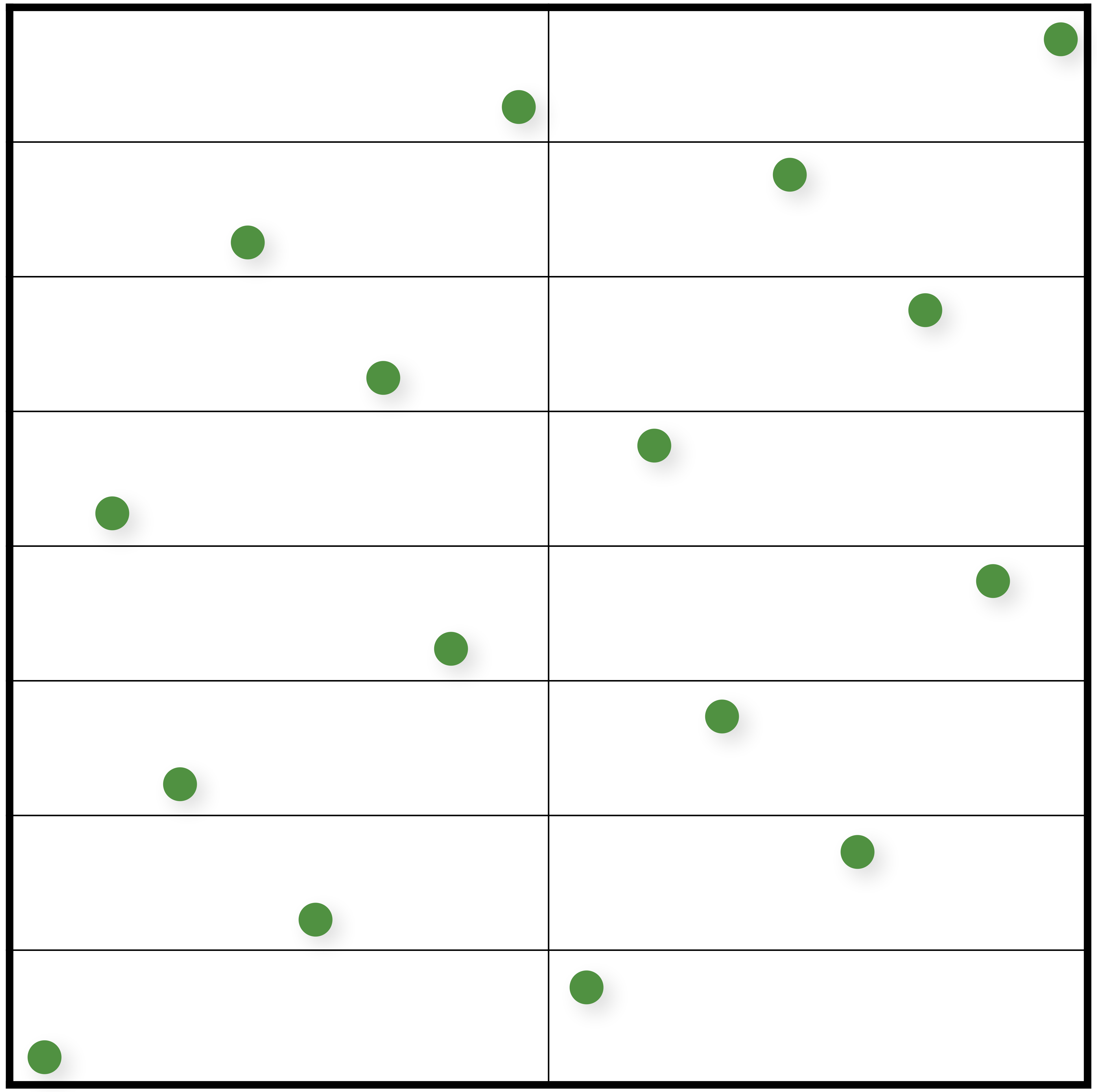
1 sample in each "elementary interval"

The Hammersley Sequence



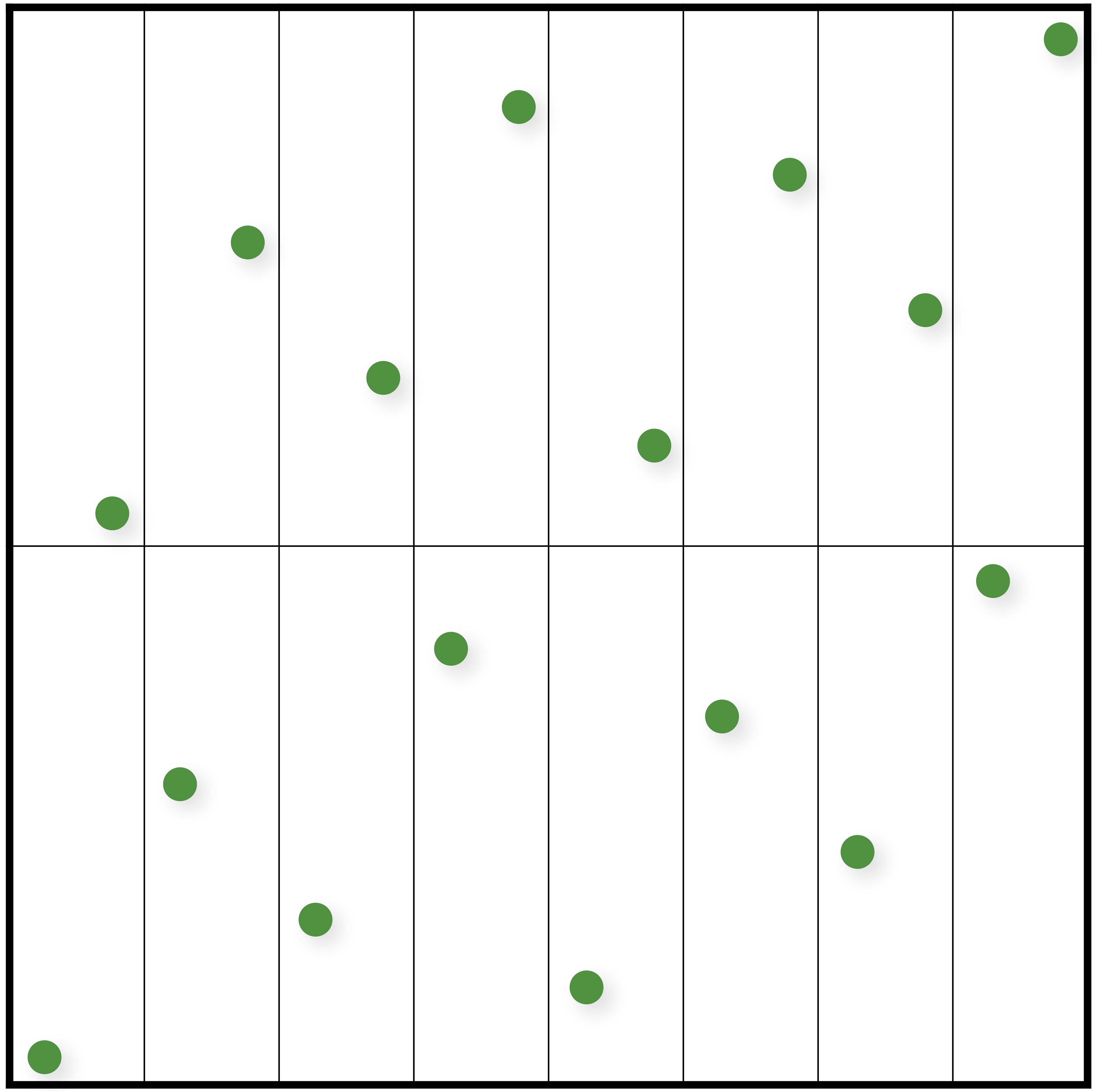
1 sample in each “elementary interval”

The Hammersley Sequence



1 sample in each "elementary interval"

The Hammersley Sequence



1 sample in each “elementary interval”

Monte Carlo (16 random samples)

Monte Carlo (16 jittered samples)

Scrambled Low-Discrepancy Sampling

More info on QMC in Rendering

S. Premoze, A. Keller, and M. Raab.
Advanced (Quasi-) Monte Carlo Methods for Image Synthesis.
In SIGGRAPH 2012 courses.