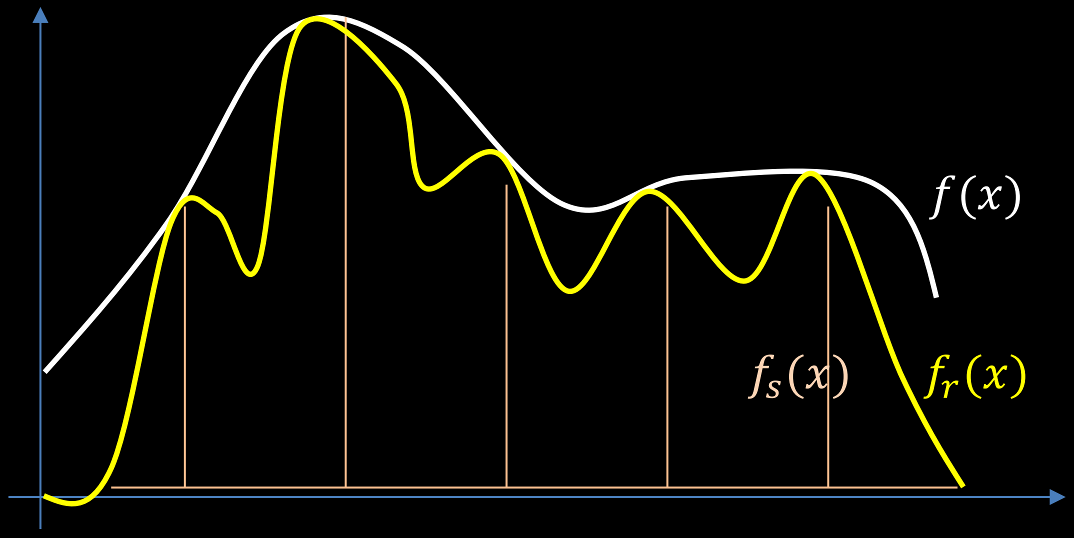
Computer Graphics

Lecture 14: Sampling I

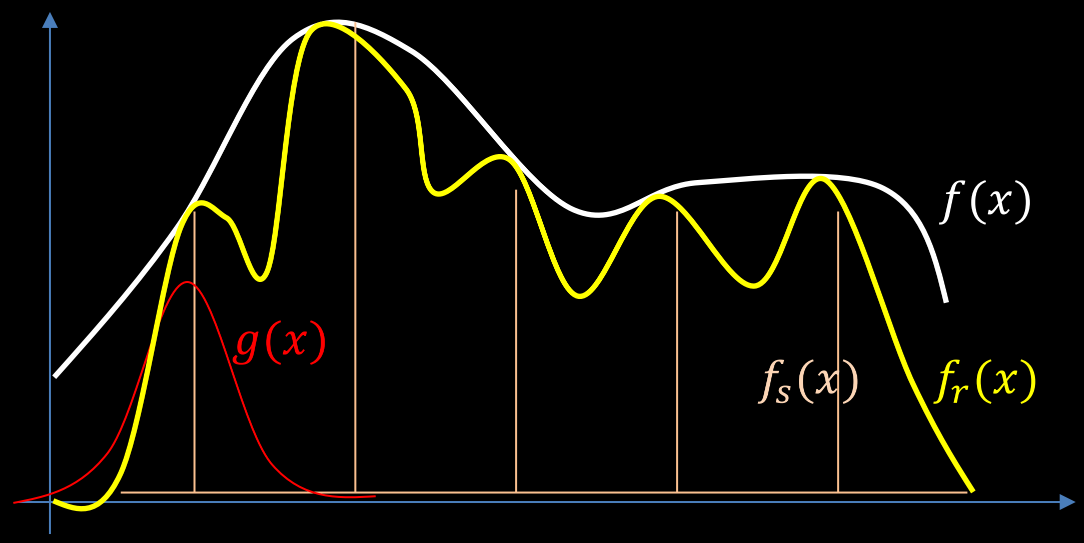
Kartic Subr

What is sampling?

Function reconstruction problem



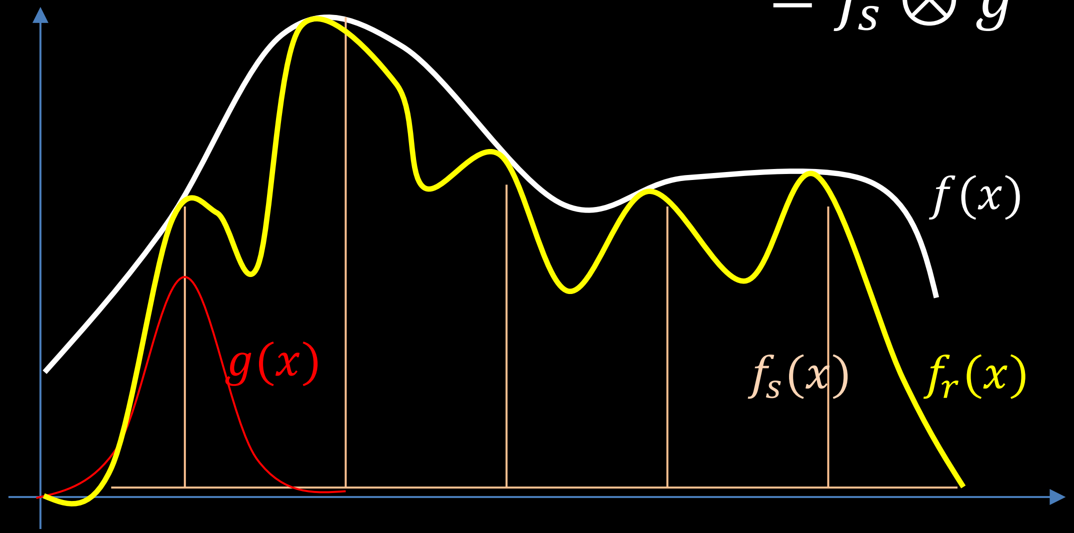
Interpolate samples using a fixed function g(x)



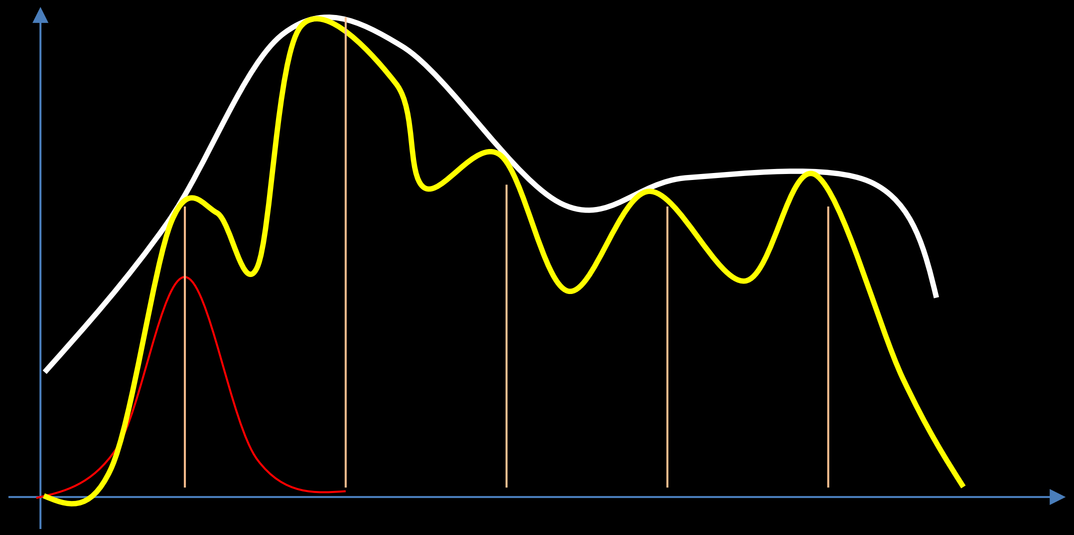
Convolution with a `reconstruction kernel'

$$f_r(x) = \int f_s(x-y)g(y)dy$$

$$= f_s \otimes g$$



How to reduce reconstruction error?



Some preliminaries: this lecture

Dirac-Delta distribution

Dirac-Delta distribution

1. Zero everywhere else

2. Unit integral

3. Identity function for convolution

Recall: 'Operate on' a vector?

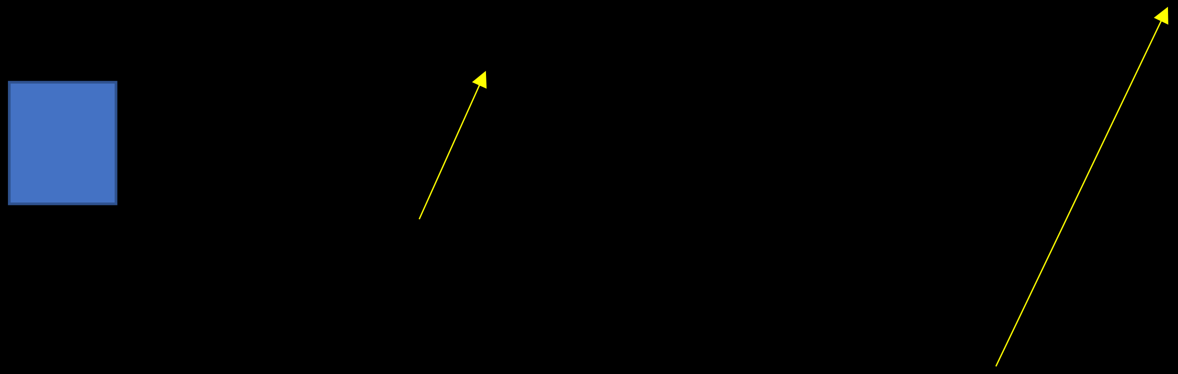
operator

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix}$$

Which vector – unaffected by operator?

?

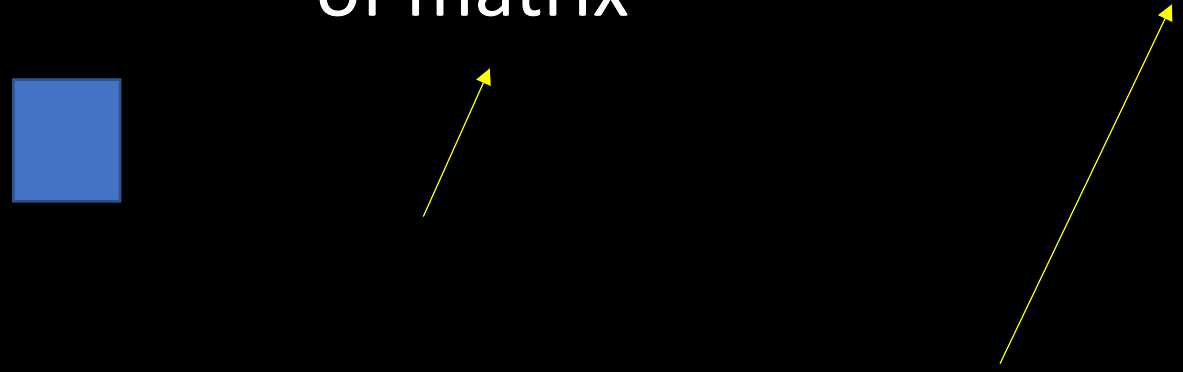
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \text{constant} \begin{pmatrix} u \\ v \end{pmatrix}$$



Which vector – unaffected by operator?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \text{constant} \begin{pmatrix} u \\ v \end{pmatrix}$$

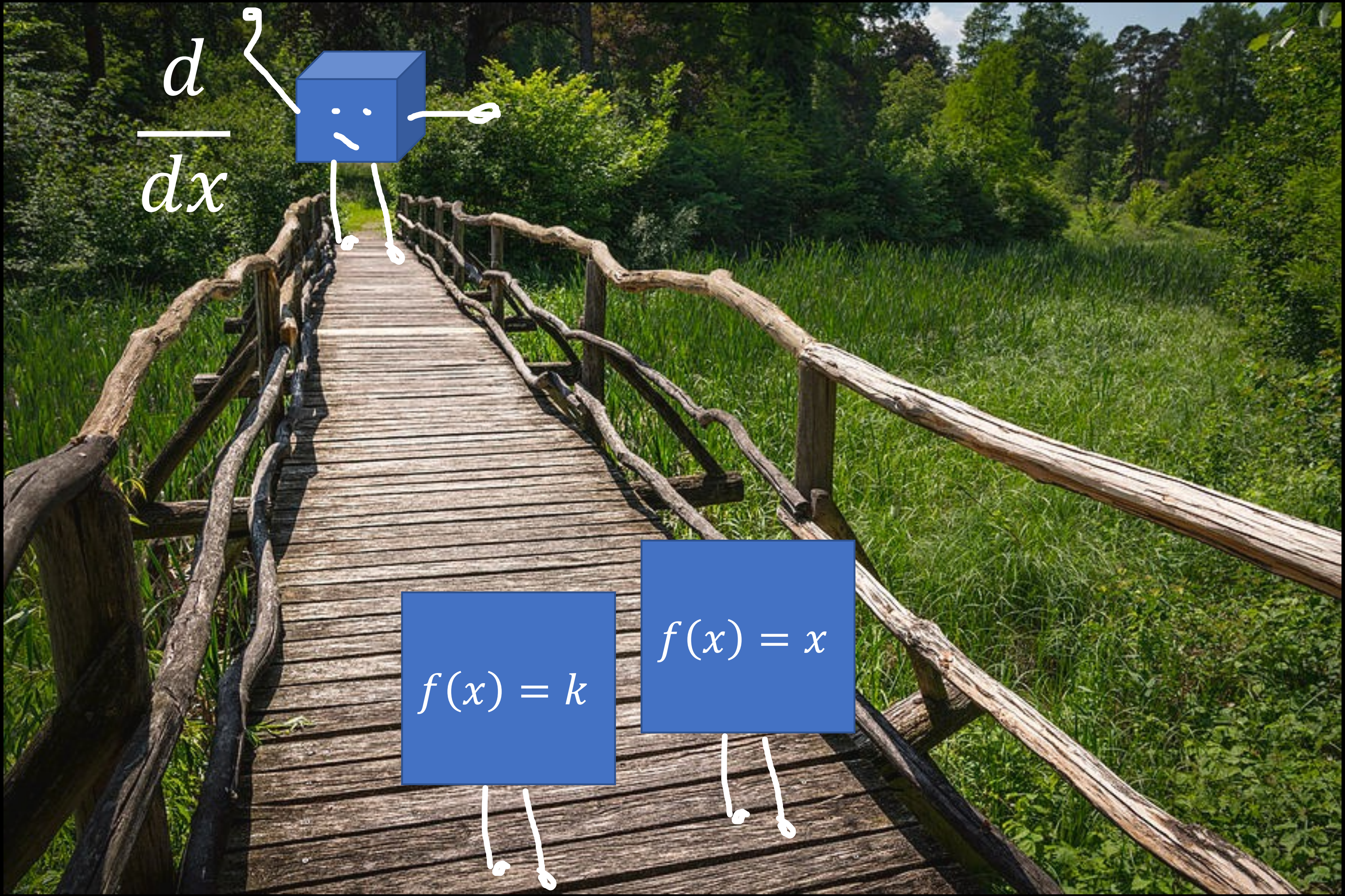
Eigenvector of matrix



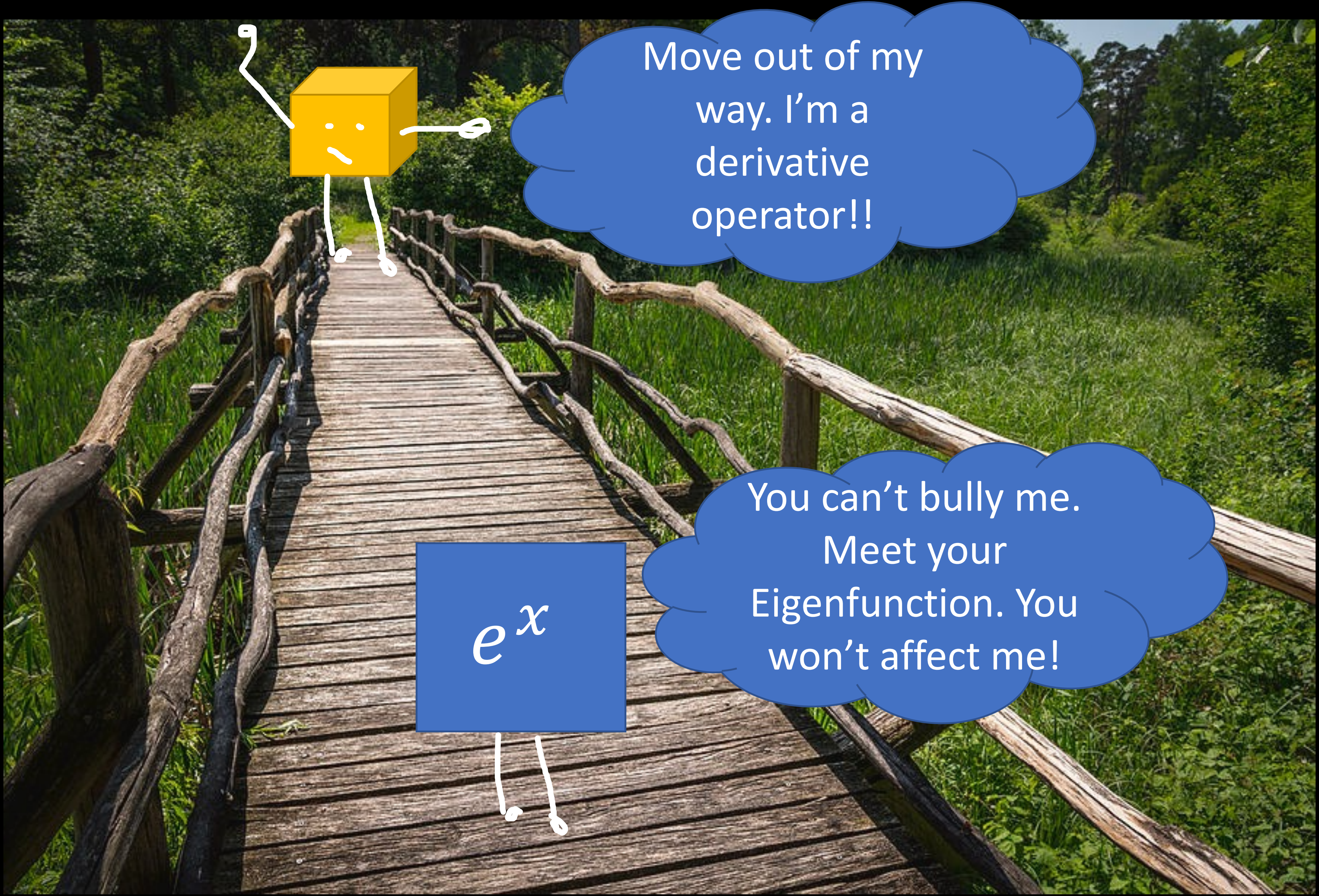
Continuous - Eigenfunctions

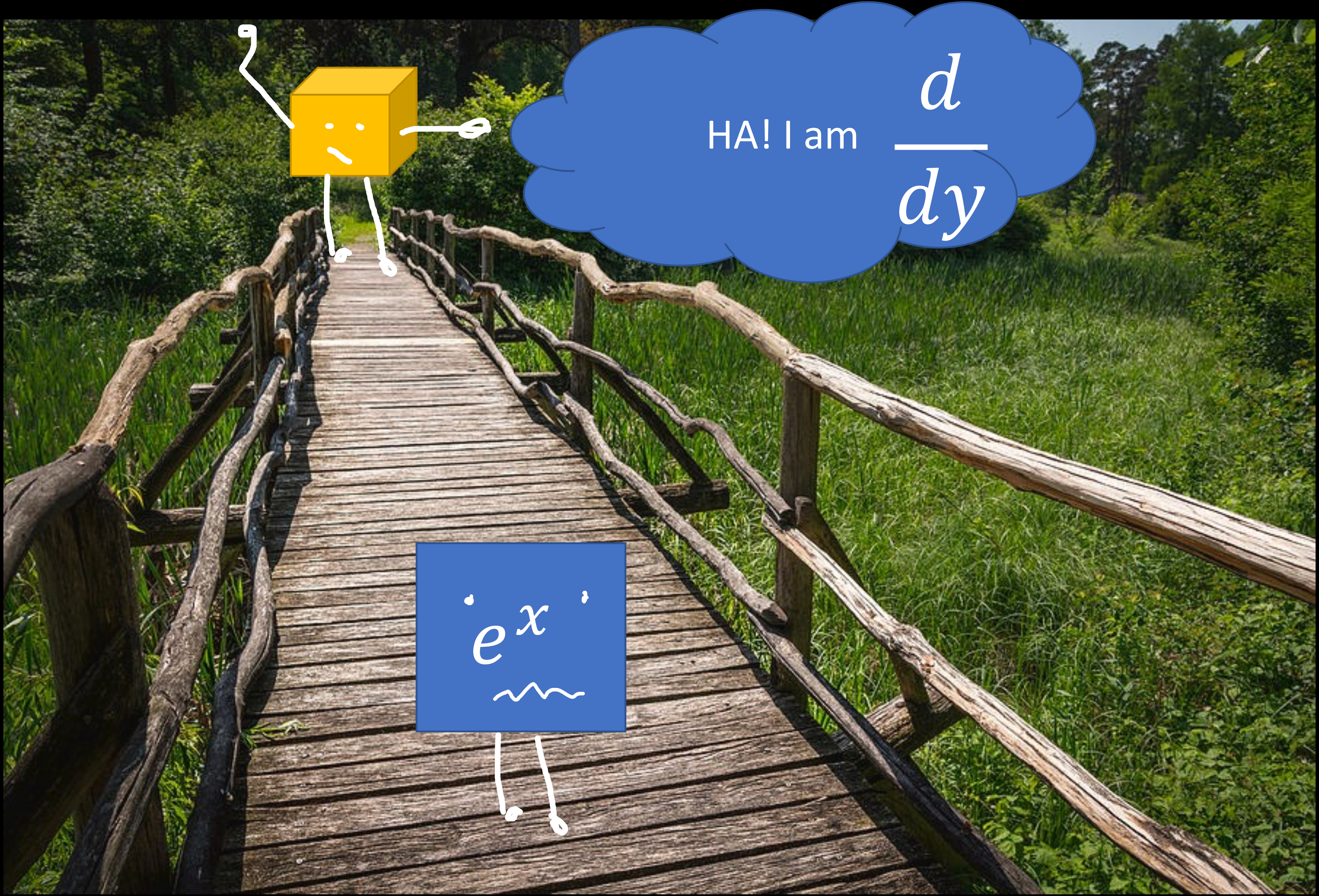
$$\frac{d}{dx}$$

Eigenfunction of differential operator?



$\frac{d}{dx}$
$f(x) = k$
$f(x) = x$


Move out of my way. I'm a derivative operator!!
e^x
You can't bully me. Meet your Eigenfunction. You won't affect me!

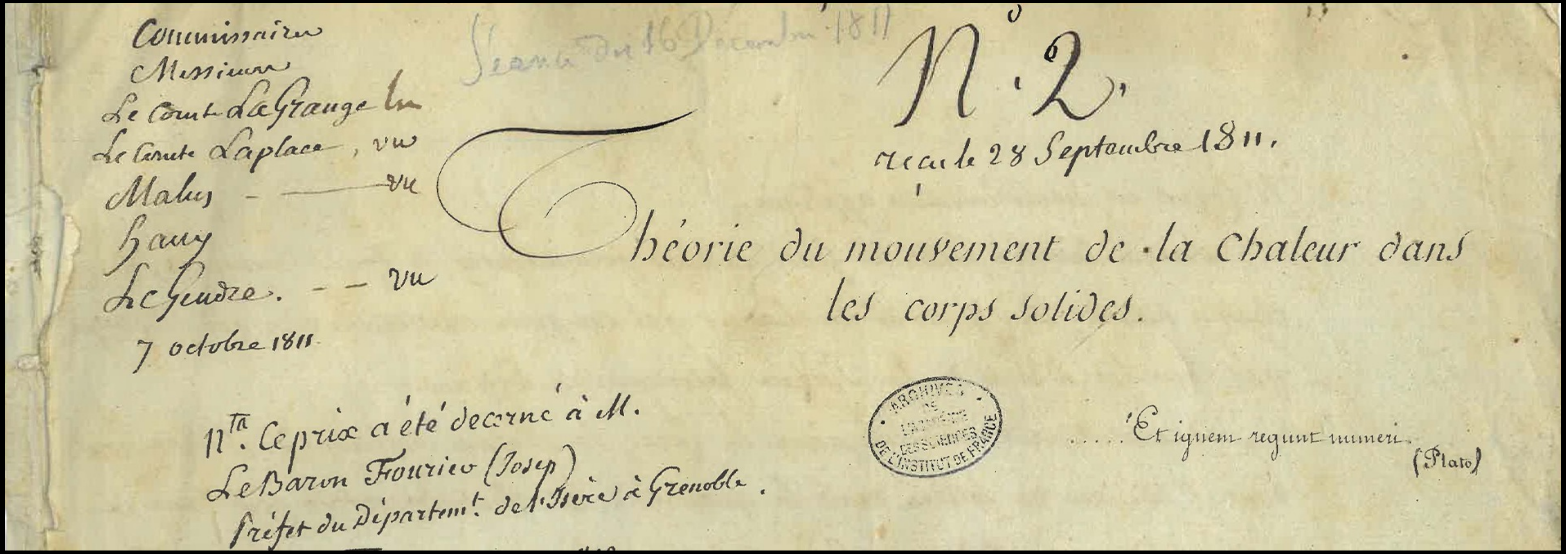

HA! I am $\frac{d}{dy}$
e^x

Fourier analysis: origin and intuition

- Eigenfunction of the differential operator

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{\lambda x} = \lambda e^{\lambda x}$$

scaling


Commissaires
Messieurs
Le Comte LaGrange lu
Le Comte Laplace, vu
Malus —— vu
Haüy
LeGendre. — — vu
7 octobre 1811

Séance du 16 Décembre 1811

N°. 2.
reçu le 28 Septembre 1811.

Théorie du mouvement de la Chaleur dans les corps solides.

N.ta Ce prix a été decerné à M.
Le Baron Fourier (Joseph)
Préfet du Départemt. de l'Isère à Grenoble.

ARCHIVES DE L'ACADÉMIE DES SCIENCES DE L'INSTITUT DE FRANCE

..... Et ignem regunt numeri. (Plato)

Joseph Fourier, 1807

Use this to solve differential equations

- Eigenfunction of the differential operator

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{\lambda x} = \lambda e^{\lambda x}$$

scaling

- differential equations -> algebraic equations

$$f(x) = \sum_{i=1}^{N} e^{\lambda_i x}, \quad \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \sum_{i=1}^{N} \lambda_i e^{\lambda_i x}$$

projection

If λ is complex, then sinusoids …

Euler's Formula

$$e^{i\phi} = \cos\phi + i\sin\phi$$

The Fourier domain

Image credits: Wikipedia

A special trigonometric series which could represent any arbitrary function

The continuous Fourier transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi \imath \omega x} \mathrm{d}x$$

Fourier domain

primal (space, time, etc.) domain

The Fourier transform: `frequency’ domain

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi \imath \omega x} \mathrm{d}x$$

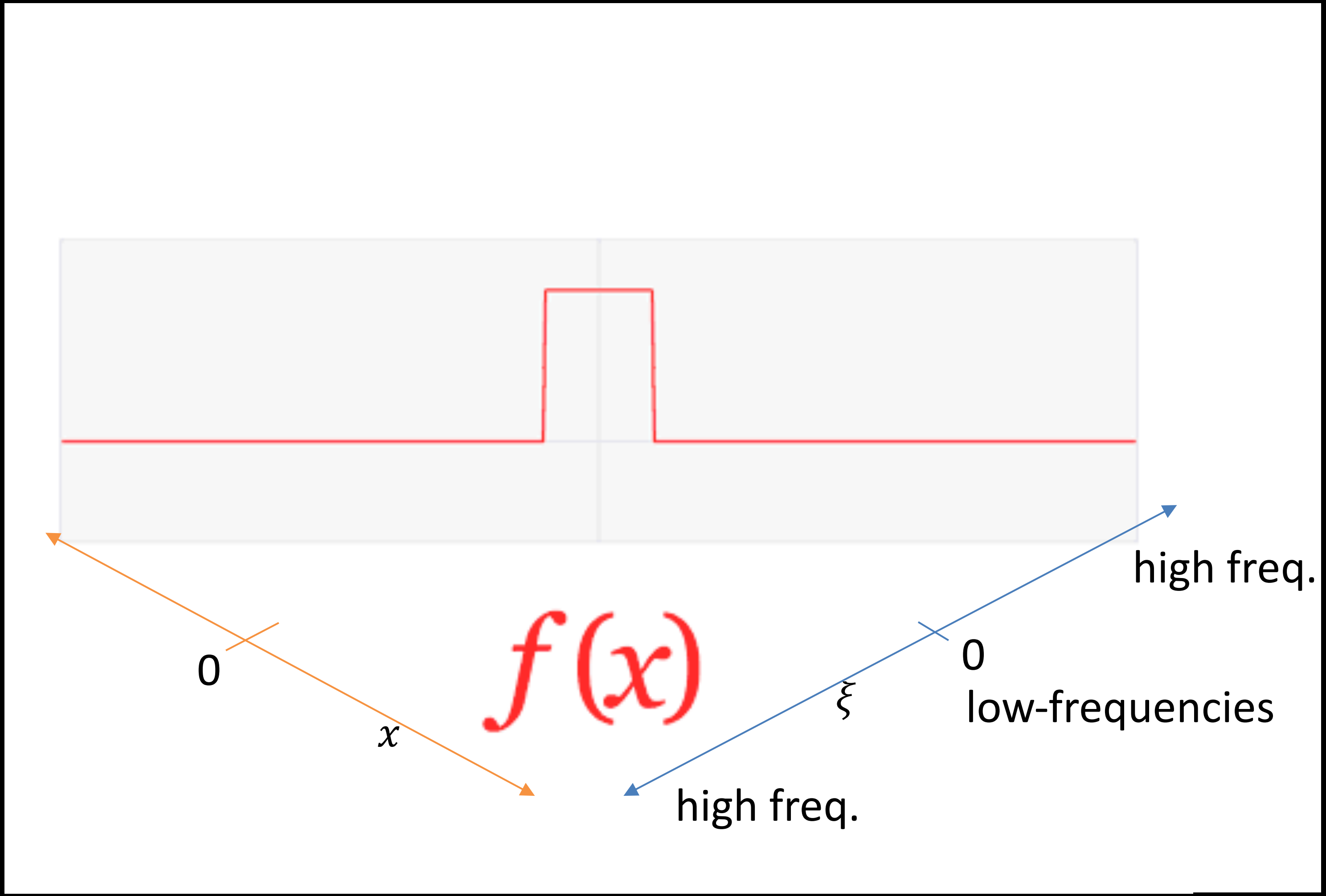
$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) \cos(2\pi \underline{\omega} x) \mathrm{d}x + \imath \int_{-\infty}^{\infty} f(x) \sin(2\pi \omega x) \mathrm{d}x$$

frequency domain

frequency

projection onto sin and cos

The Fourier Transform

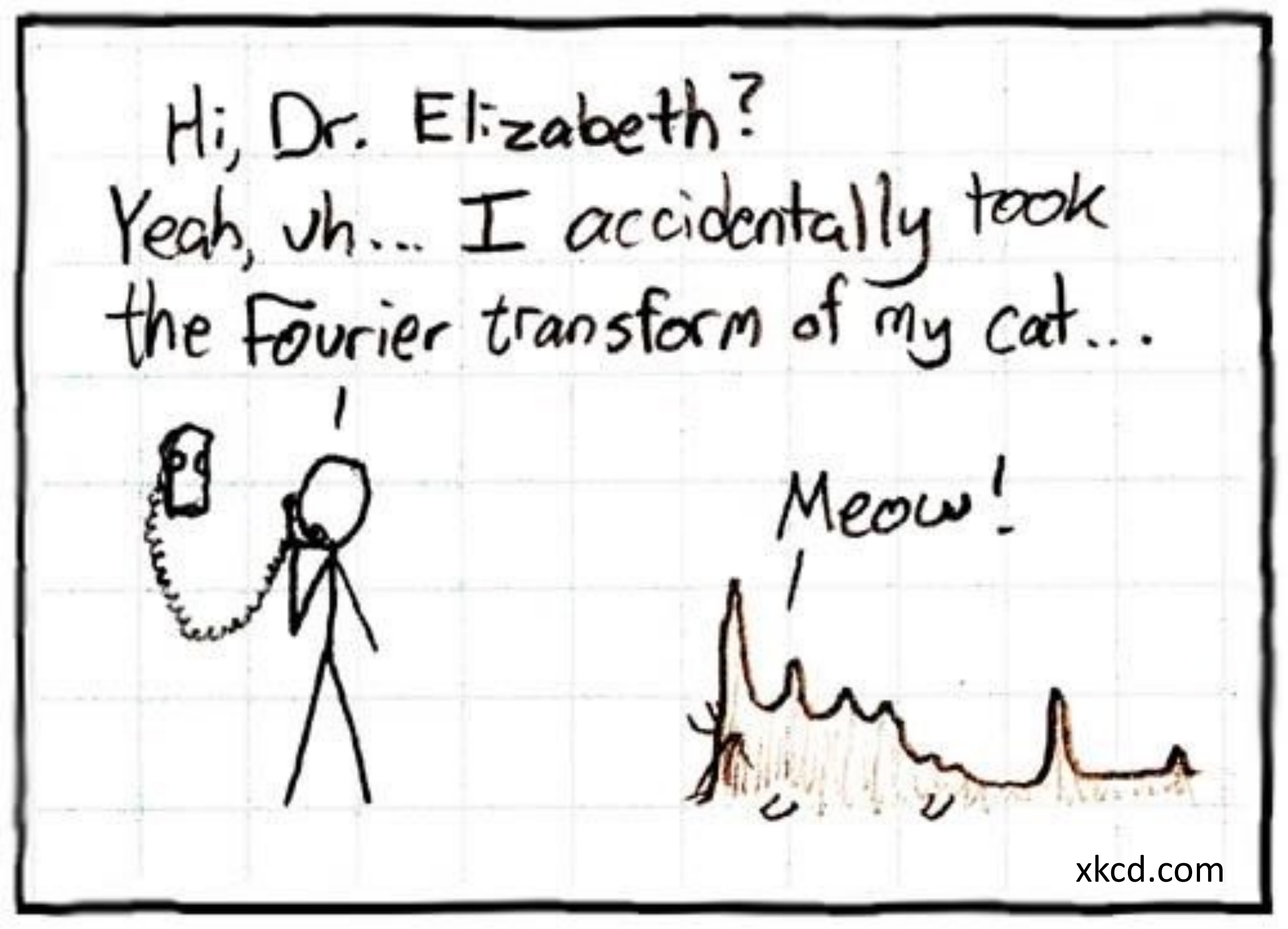


Fourier "duals"

$f(x) \Longleftrightarrow F(\xi)$

$f(x)$	$F(\xi)$
$\mathrm{rect}(ax)$	$\frac{1}{\|a\|} \cdot \mathrm{sinc}\left(\frac{\xi}{a}\right)$
$\mathrm{sinc}(ax)$	$\frac{1}{\|a\|} \cdot \mathrm{rect}\left(\frac{\xi}{a}\right)$
$\mathrm{sinc}^2(ax)$	$\frac{1}{\|a\|} \cdot \mathrm{tri}\left(\frac{\xi}{a}\right)$
$\mathrm{tri}(ax)$	$\frac{1}{\|a\|} \cdot \mathrm{sinc}^2\left(\frac{\xi}{a}\right)$
$e^{-ax}u(x)$	$\frac{1}{a+2\pi i\xi}$
$e^{-\alpha x^2}$	$\sqrt{\frac{\pi}{\alpha}} \cdot e^{-\frac{(\pi\xi)^2}{\alpha}}$
$e^{-a\|x\|}$	$\frac{2a}{a^2+4\pi^2\xi^2}$
$\mathrm{sech}(ax)$	$\frac{\pi}{a}\,\mathrm{sech}\left(\frac{\pi^2}{a}\xi\right)$
$e^{-\frac{a^2x^2}{2}}H_n(ax)$	$\frac{\sqrt{2\pi}(-i)^n}{a}e^{-\frac{2\pi^2\xi^2}{a^2}}H_n\left(\frac{2\pi\xi}{a}\right)$

What can you take the Fourier transform of?



A single sample:

$$f(x) = \delta(x - x_k)$$

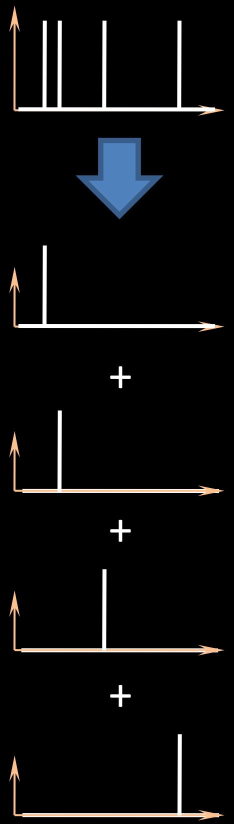
$$\hat{f}(\omega) = e^{-2\pi \imath x_k \omega}$$

phase

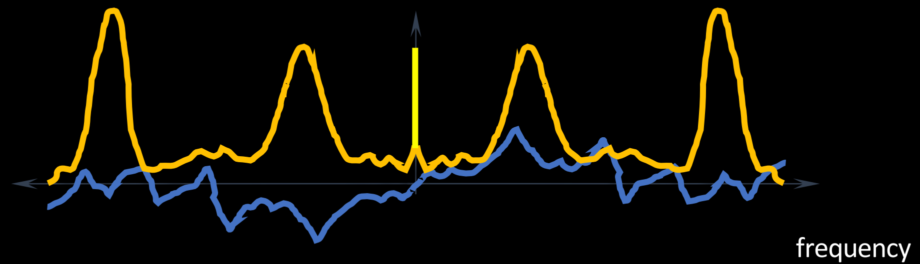
amplitude = 1

$$\hat{f}(\omega) = \cos(2\pi \imath x_k \omega) + \imath \sin(2\pi \imath x_k \omega)$$

sampling function = sum of Dirac deltas



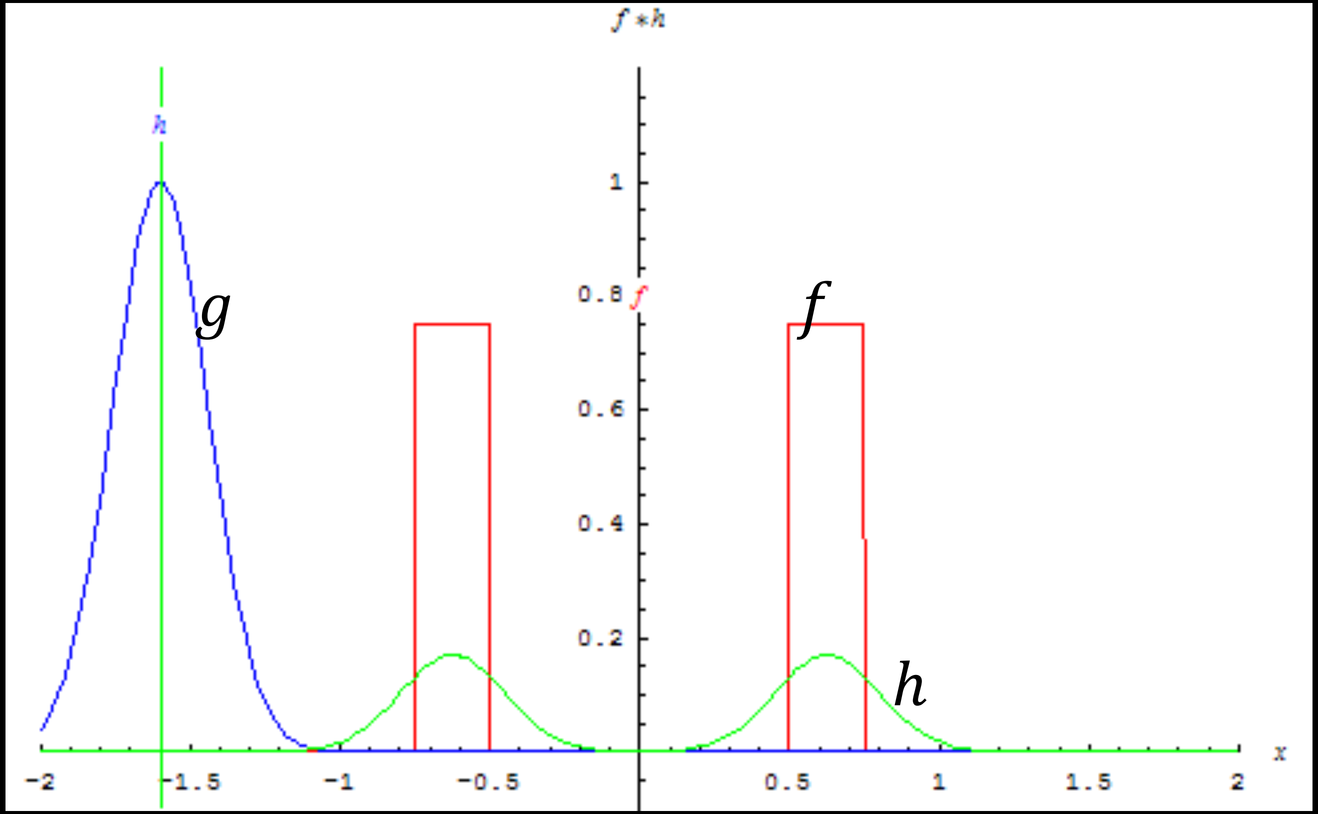
Fourier spectrum of the sampling function



$$S(x) = \sum_{k=1}^{N} \delta(x - x_k)$$

$$\hat{S}(\omega) = \sum_{k=1}^{N} e^{-2\pi \imath x_k \omega}$$

Remember convolution?



$$h(x) = \int f(x-y)g(y)dy$$

$$h(x) = f(x) \otimes g(x)$$

Fourier Transform of Convolution ?

$$h(x) = f(x) \otimes g(x)$$

Fourier Transform of Convolution ?

$$h(x) = f(x) \otimes g(x)$$

$$\mathcal{F}(h(x)) = \mathcal{F}(f(x) \otimes g(x))$$

Fourier Transform of Convolution ?

$$h(x) = f(x) \otimes g(x)$$

$$\mathcal{F}(h(x)) = \mathcal{F}(f(x) \otimes g(x))$$

$$H(\xi) = F(\xi)\, G(\xi)$$

Convolution theorem

$$\mathcal{F}\left(f(x) \otimes g(x)\right) = F(\xi)\, G(\xi)$$

Fourier transform of a convolution

product of Fourier transformed functions

Alternative way to calculate convolutions

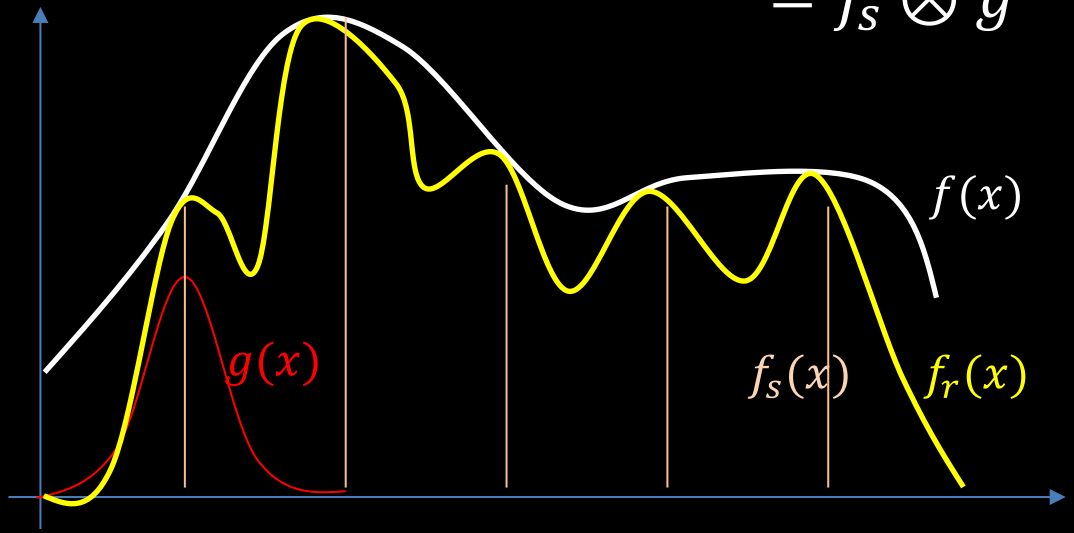
$$h(x) = \int f(x-y)g(y)dy$$

Fast Fourier Transform 1. Obtain Fourier transforms F and G
2. Multiply, so H = F.G
Fast Fourier Transform 3. Take the inverse Fourier transform of H
4. h = H^{-1}

What if we apply the Fourier transform?

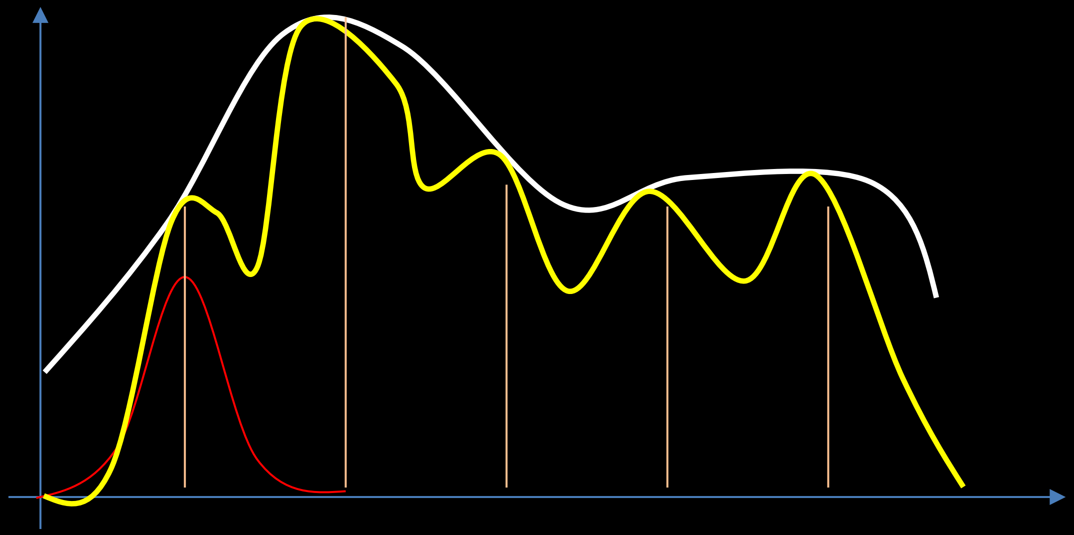
$$f_r(x) = \int f_s(x-y)g(y)dy$$

$$= f_s \otimes g$$

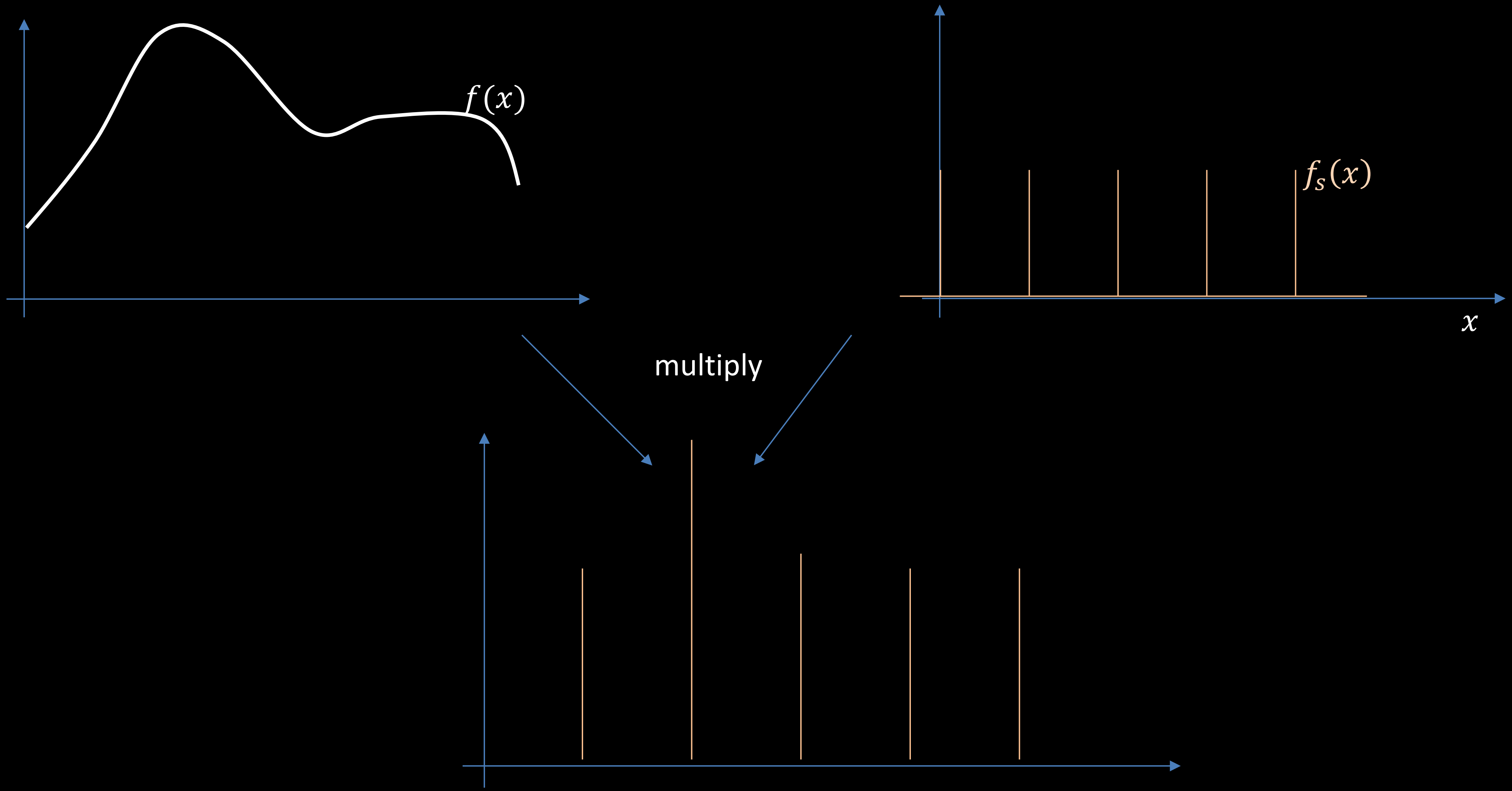


$$F_r(\xi) = F_s(\xi)\,G(\xi)$$

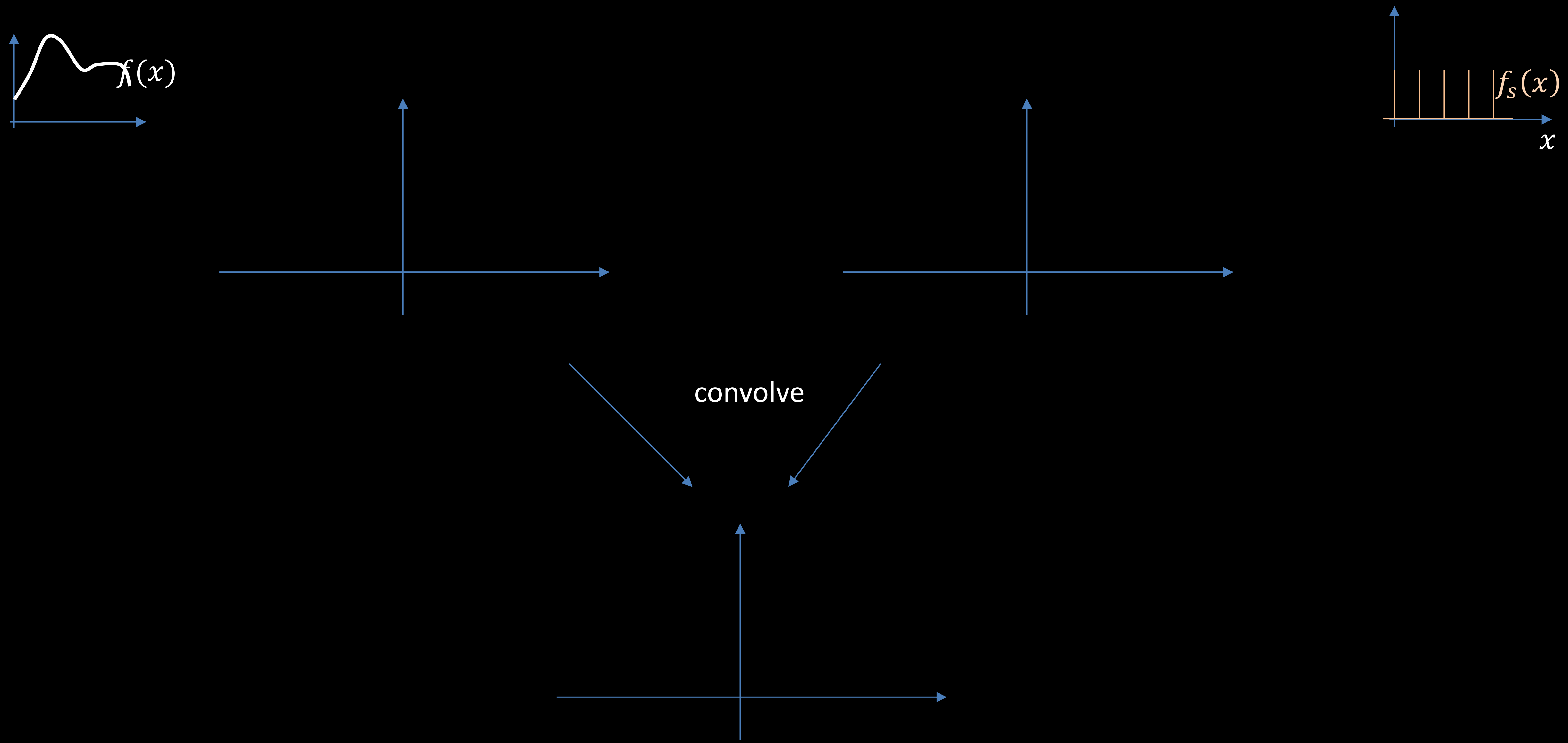
How to assess sampling and reconstruction error?



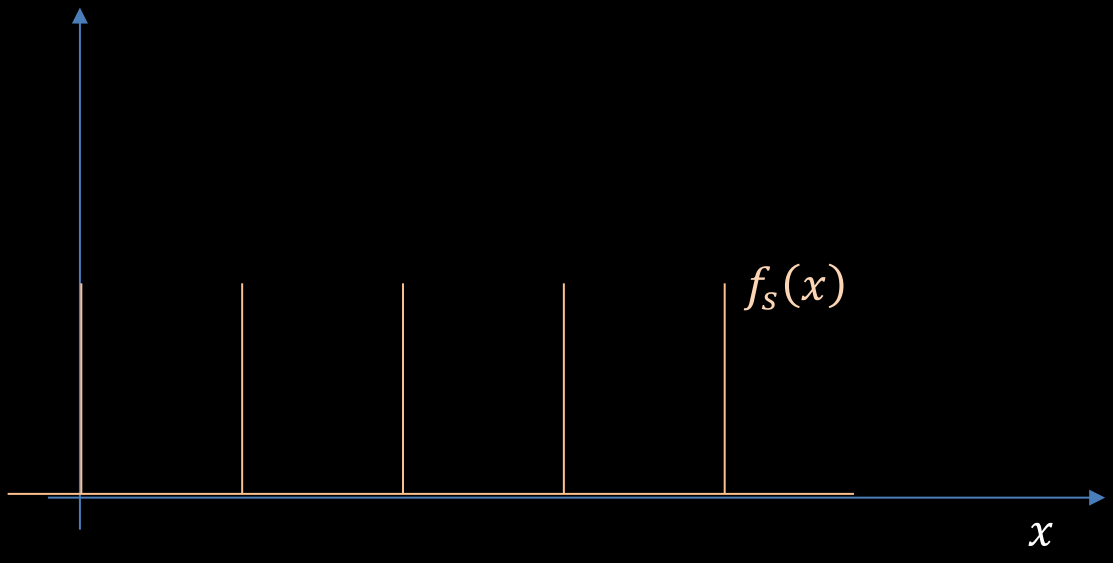
Focus on the sampling operation first:

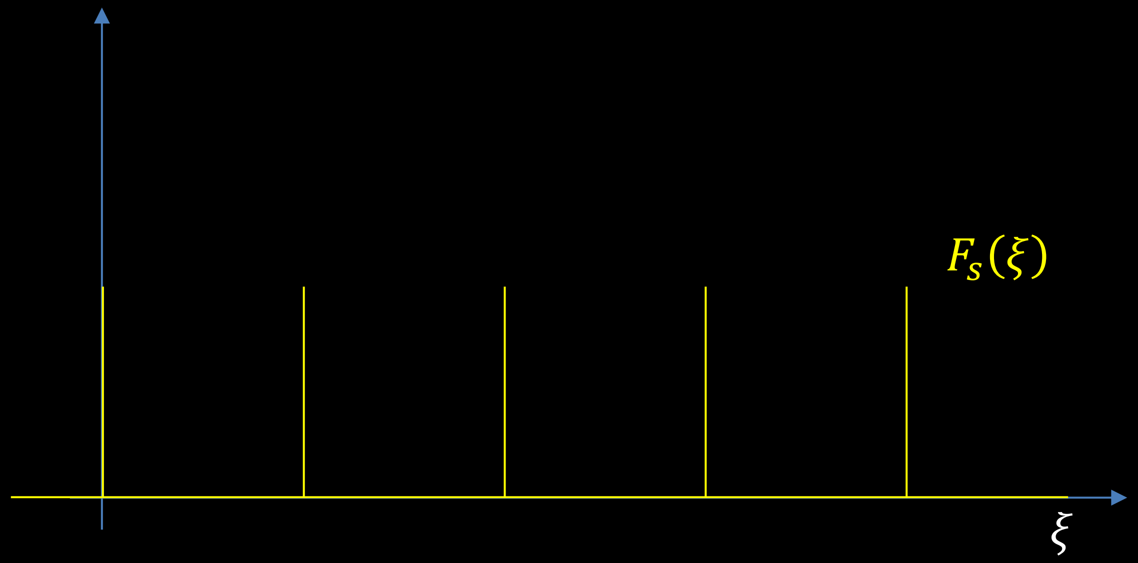


What are these in the Fourier domain?

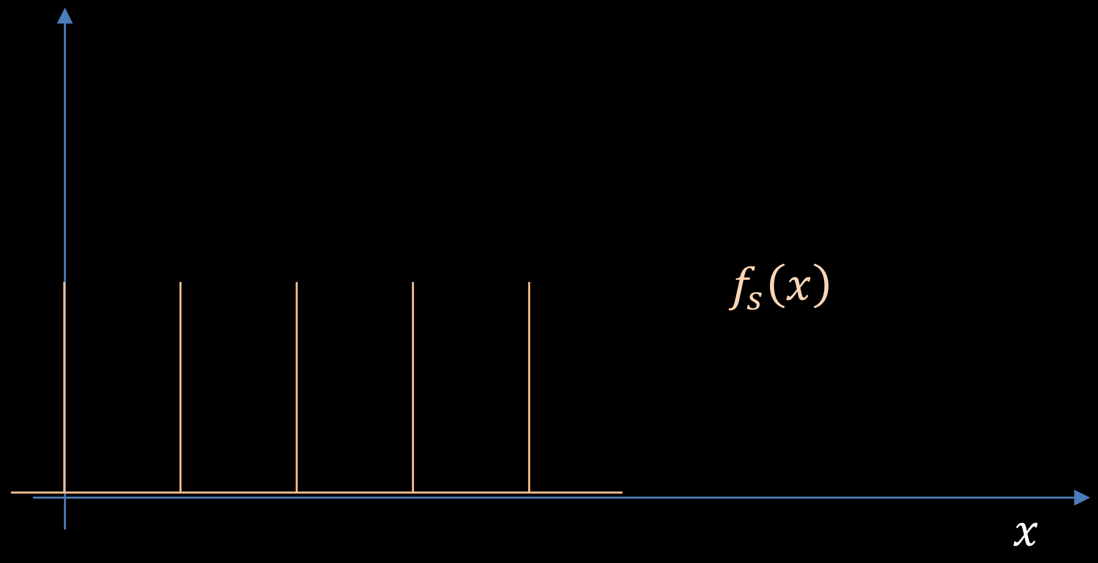


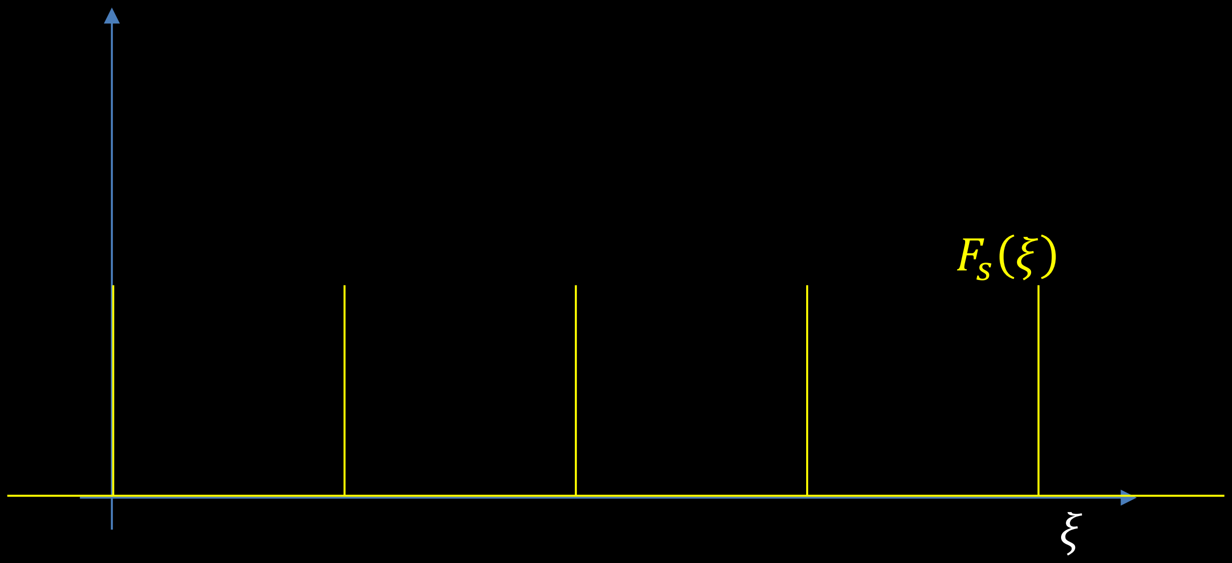
Intuition: Sampling function



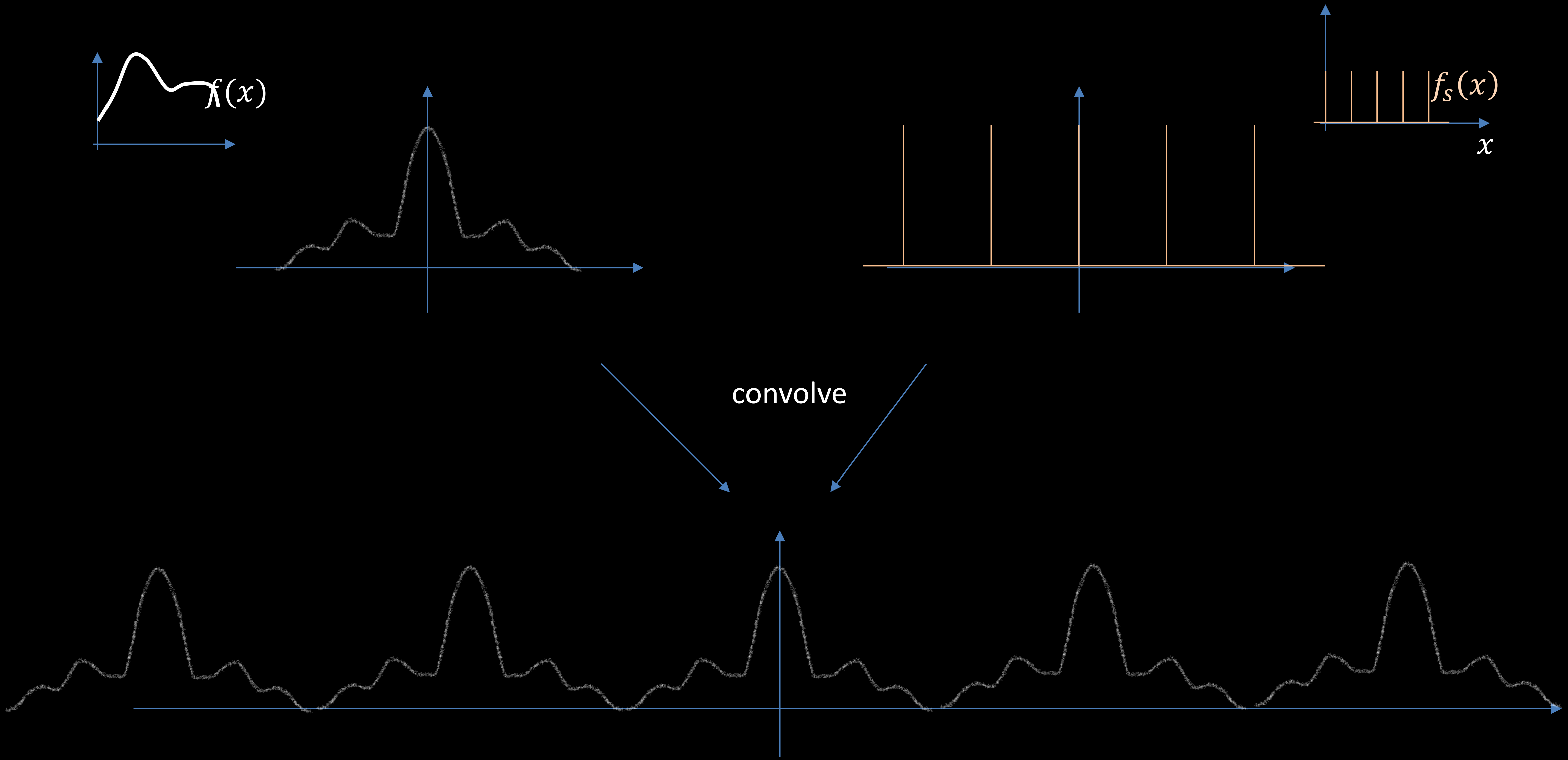


Intuition: Sampling function

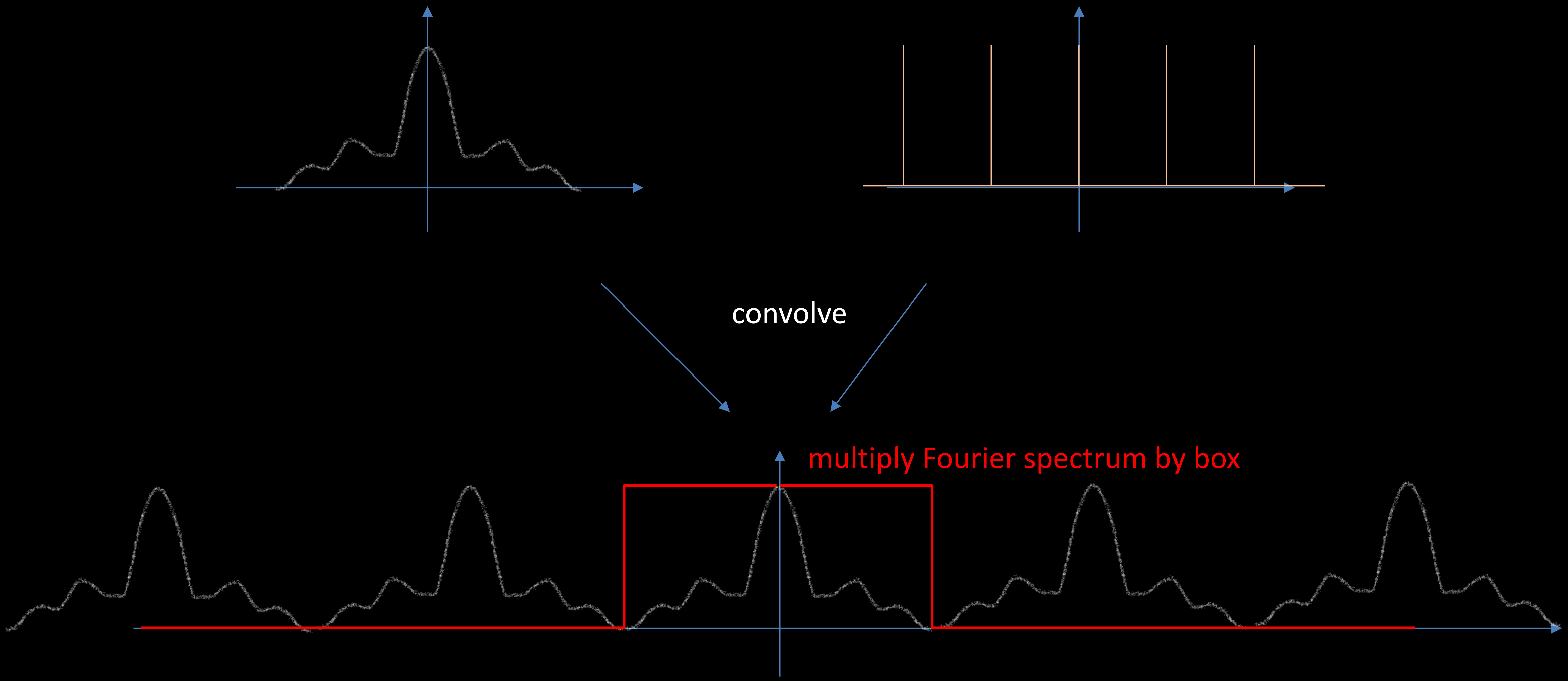




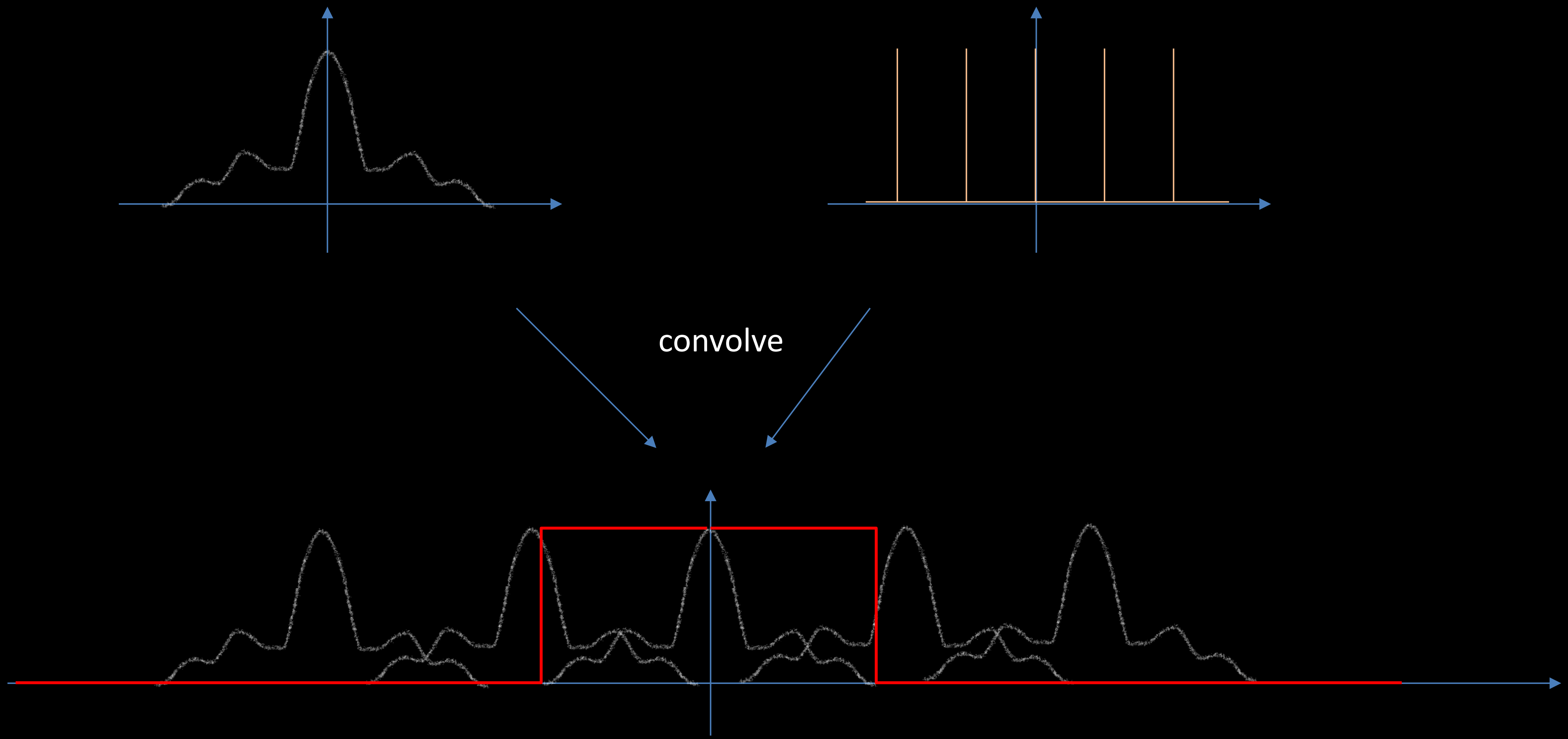
Sampling in the Fourier Domain



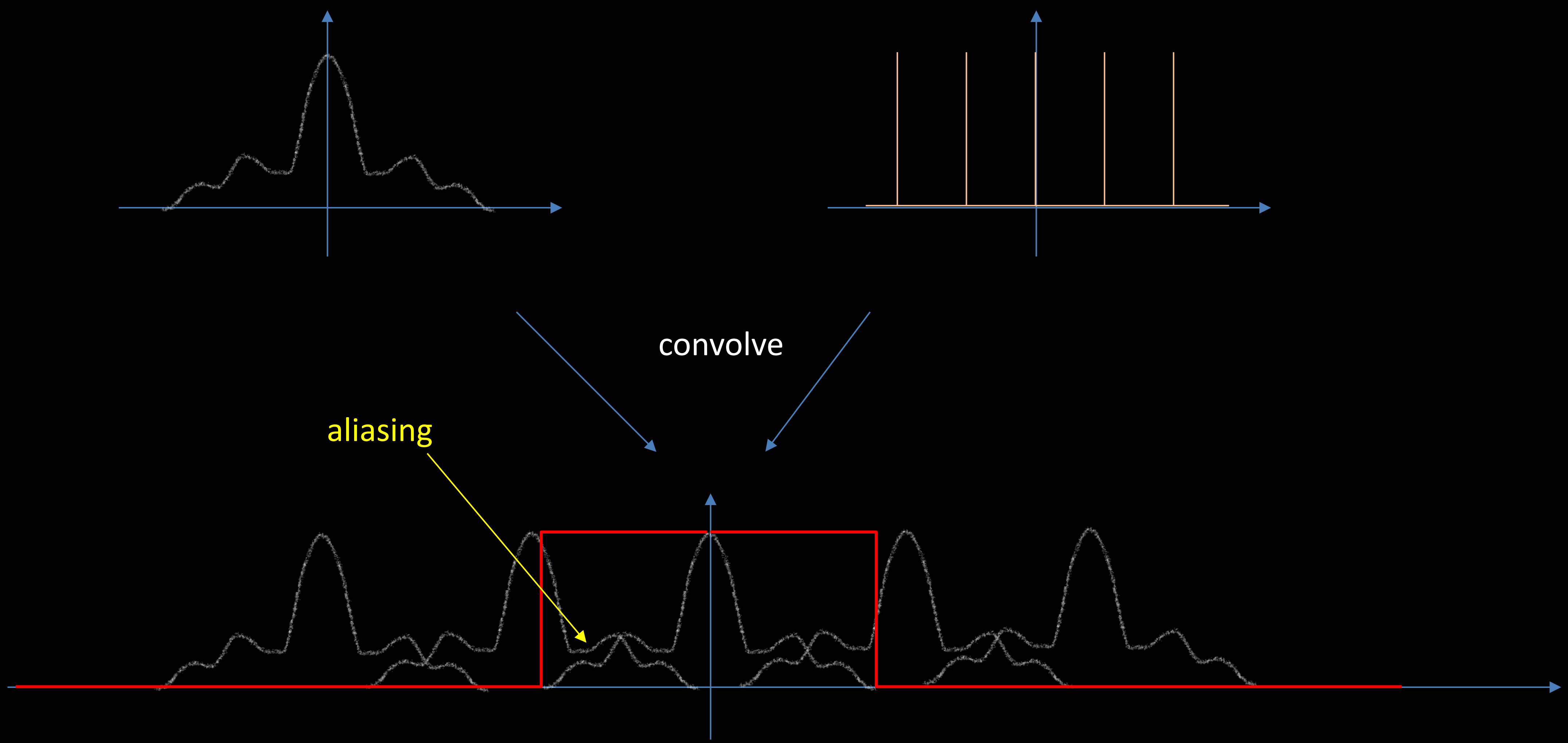
How to remove aliases?



Sparse sampling (squeezed in Fourier domain)



Sparse sampling (squeezed in Fourier domain)

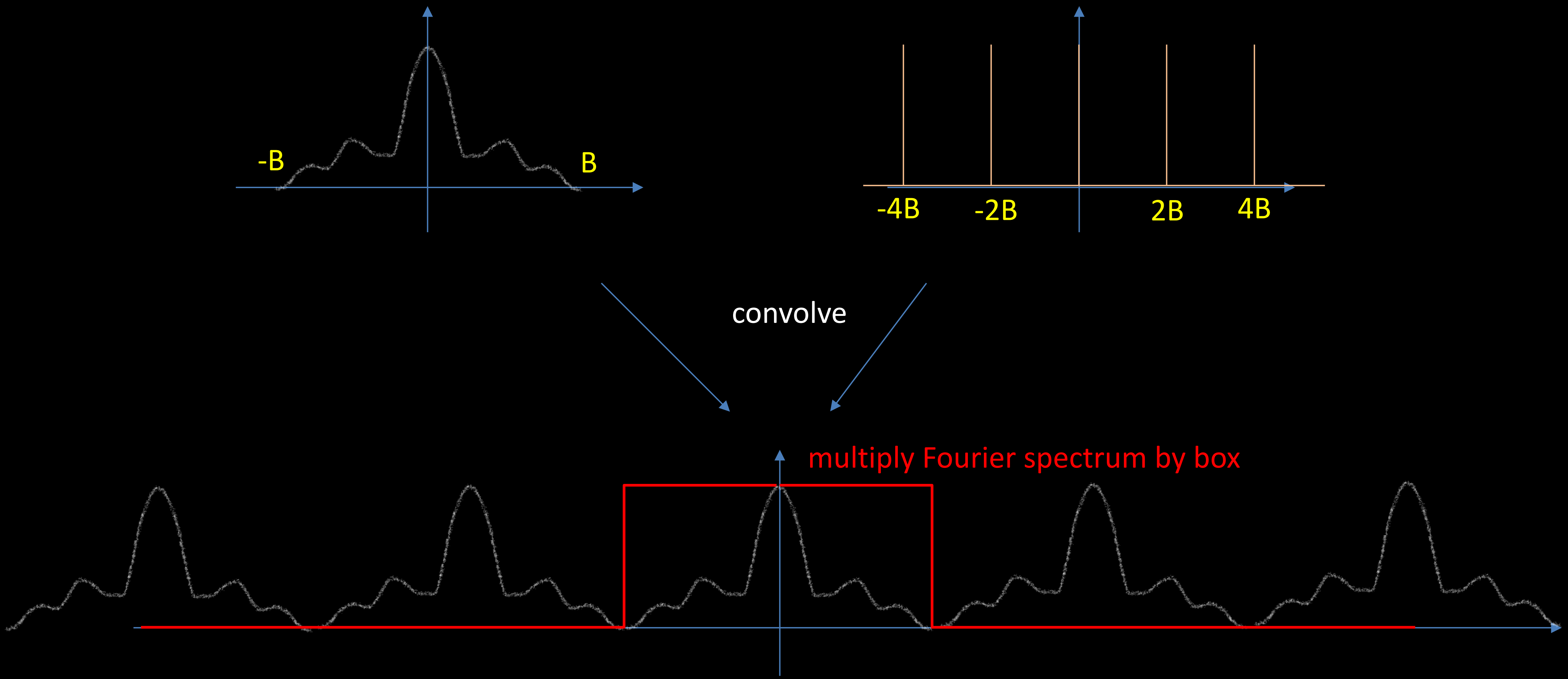


Nyquist-Shannon Sampling theorem

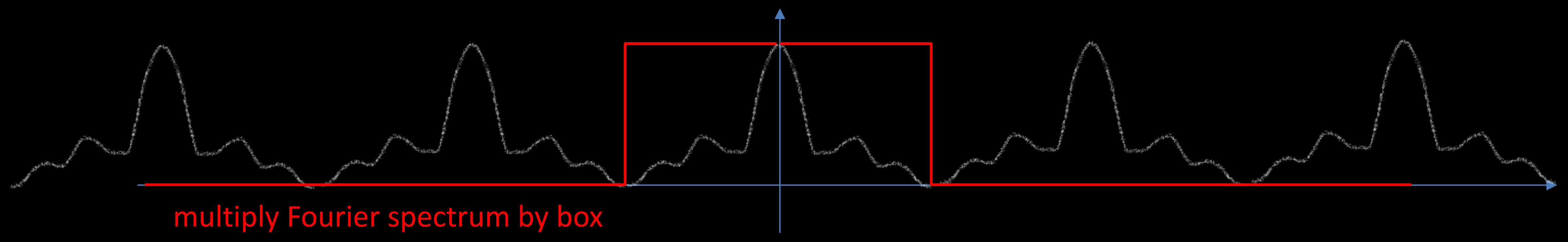
If a function $x(t)$ contains no frequencies higher than B hertz,

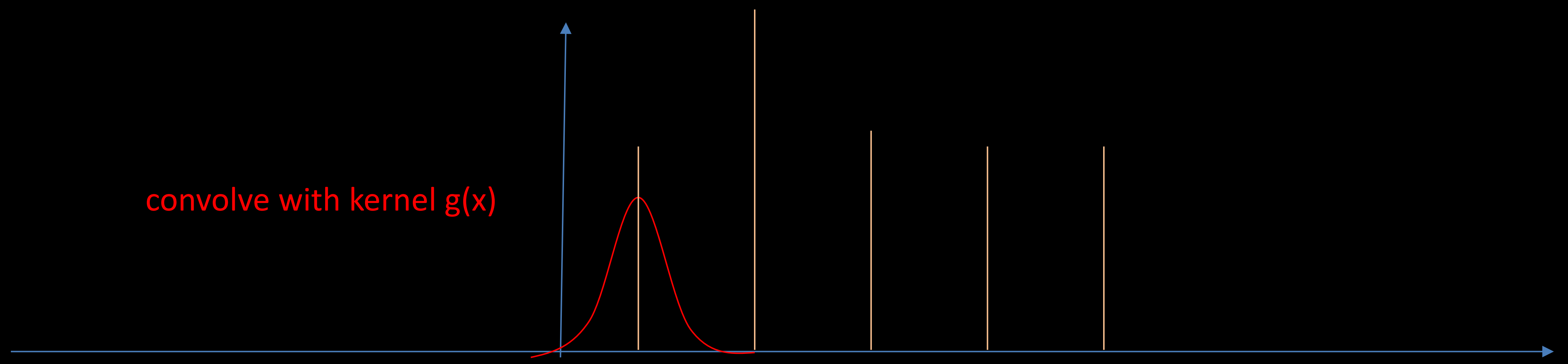
it is completely determined by giving its ordinates at a series of points spaced $1/(2B)$ seconds apart.

Multiplication by a box in the Fourier (frequency) domain…

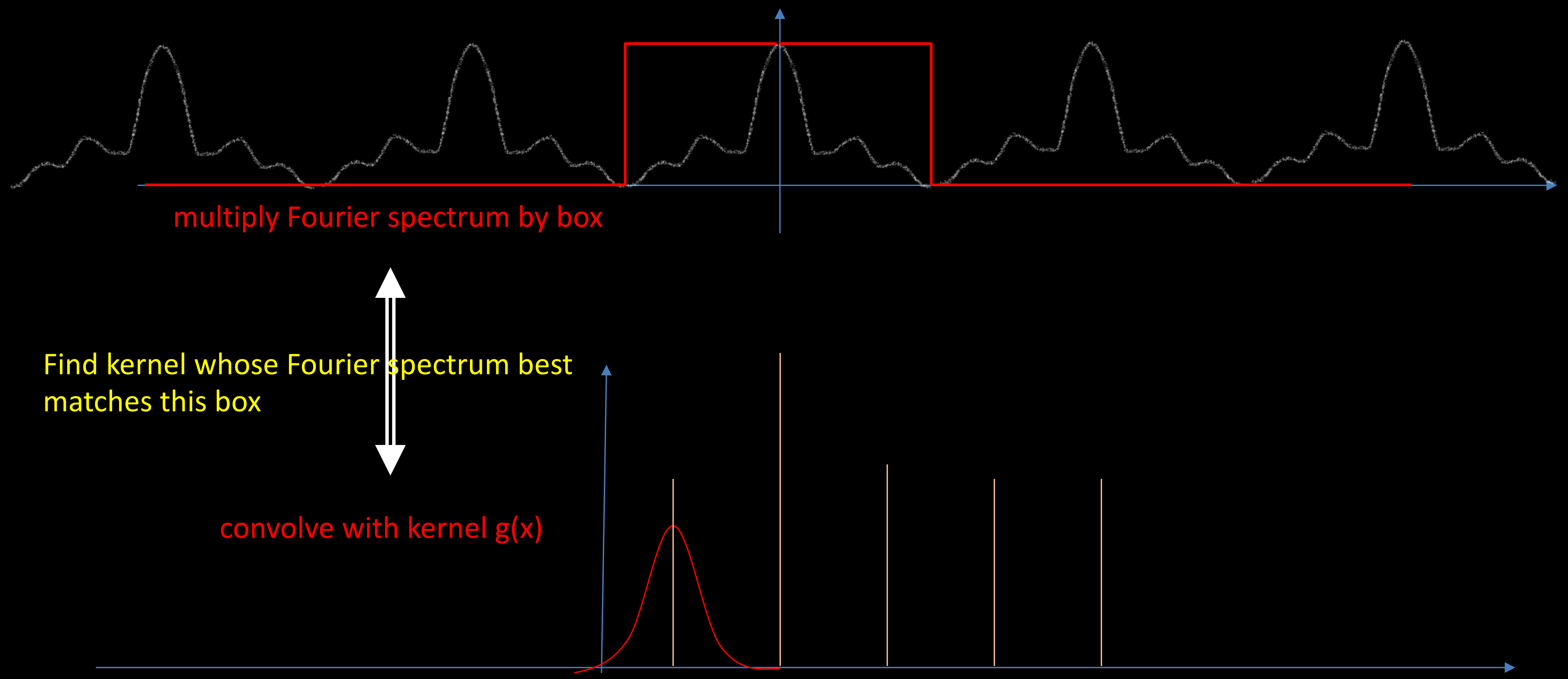


... convolution with a reconstruction kernel (in x)





Is a convolution with a reconstruction kernel (in the primal, or x)



To be continued …